ACTEX Learning

Study Manual for Exam MAS-I

8th Edition 2nd Printing

Ambrose Lo, PhD, FSA, CERA

A CAS Exam

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ACTEX Learning

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QUESTION 19 OF 704 | Question # | Go! | Prev | Next

Question — Difficulty: Advanced

An airport purchases an insurance policy to offset costs associated with excessive amounts of snowfall. The insurer pays the airport 300 for every full ten inches of snow in excess of 40 inches, up to a policy maximum of 700.

The following table shows the probability function for the random variable X of annual (winter season) snowfall, in inches, at the airport.

Inches	[0,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	[90,inf)
Probability	0.06	0.18	0.26	0.22	0.14	0.06	0.04	0.04	0.00

Calculate the standard deviation of the amount paid under the policy.

Possible Answers

A 134 | ✓ 235 | X 271 | D 313 | E 352

Help Me Start

Find the probabilities for the four possible payment amounts: 0, 300, 600, and 700.

Solution

With the amount of snowfall as X and the amount paid under the policy as Y, we have

y	$f_Y(y) = P(Y = y)$
0	$P(Y=0) = P(0 \le X < 50) = 0.72$
300	$P(Y=300) = P(50 \le X < 60) = 0.14$
600	$P(Y=600) = P(60 \le X < 70) = 0.06$
700	$P(Y=700) = P(Y \ge 70) = 0.08$

The standard deviation of Y is $\sqrt{E(Y^2) - [E(Y)]^2}$.

$$E(Y) = 0.14 \times 300 + 0.06 \times 600 + 0.08 \times 700 = 134$$

$$E(Y^2) = 0.14 \times 300^2 + 0.06 \times 600^2 + 0.08 \times 700^2 = 73400$$

$$\sqrt{E(Y^2) - [E(Y)]^2} = \sqrt{73400 - 134^2} = 235.465$$

Common Questions & Errors

Students shouldn't overthink the problem with fractional payments of 300. Also, account for probabilities in which payment cap of 700 is reached.

In these problems, we must distinguish between the REALT RV (how much snow falls) and the PAYMENT RV (when does the insurer pay)?. The problem states "The insurer pays the airport 300 for every full ten inches of snow in excess of 40 inches, up to a policy maximum of 700 ." So the insurer will not start paying UNTIL AFTER 10 full inches in excess of 40 inches of snow is reached (say at 50+ or 51). In other words, the insurer will pay nothing if X<50.

Rate this problem | Excellent | Needs Improvement | Inadequate

Quickly access the Hub for additional learning.

Flag problems for review, record notes, and get instructor support.

View difficulty level.

Helpful strategies to get you started.

Full solutions with detailed explanations to deepen your understanding.

Commonly encountered errors.

Rate a problem or give feedback.

Contents

Preface

NOTE TO STUDENTS

Please read this preface carefully. It contains very important information that will help you navigate this study manual and Exam MAS-I smoothly! 👍

1 About Exam MAS-I

Exam Theme

In 2018, the Casualty Actuarial Society (CAS) added a considerable amount of material on predictive analytics to its Associateship curriculum in view of the growing relevance of this discipline to actuarial work, especially in property and casualty (P&C) insurance. The most significant changes were the introduction of two exams bearing **"Modern Actuarial Statistics (MAS)"** in their title: **Exam MAS-I** and **Exam MAS-II**, both of which serve to enhance the predictive modeling and data science skill set of P&C actuaries in this day and age. This study manual prepares you adequately for Exam MAS-I, but also paves the way for Exam MAS-II.

As its name suggests, you will see a lot of "statistics" in Exam MAS-I, some classical and some "modern." Besides doing probability calculations and classical statistical inference (e.g., parameter estimation and hypothesis testing), you will also work with statistical models and study some contemporary predictive modeling techniques. You will learn the general tools available for constructing and evaluating predictive models (e.g., training/test set split, cross-validation), and the technical details of specific types of models (e.g., linear models, generalized linear models, shrinkage methods). After taking (and, with the use of this study manual, passing!) MAS-I, you will gain the foundational knowledge behind the modeling process.

According to the current syllabus (known as the "content outline" since Fall 2023) available from

https://www.casact.org/exam/exam-mas-i-modern-actuarial-statistics-i,

the exam consists of three domains (or sections), covering miscellaneous topics in applied probability, mathematical statistics, and statistical modeling:

(As a conservative estimate, you will need at least *four months* of intensive study to master the material in this exam.)

Domain	**Domain Weight**
A. Probability Models (Stochastic Processes and Survival Models)	20-30%
B. Statistics	20-30%
C. Extended Linear Models [**MOST IMPORTANT!**]	45-55%

As you can see, each of Domains A and B accounts for about 25% of the exam, and Domain C alone occupies about 50%.

(The exam syllabus used to have Domain D: Time Series with Constant Variance, but it has been moved to Exam MAS-II effective from Fall 2023. MAS-I students now have less to study...a bit less!)

Exam Style

With effect from Fall 2023, Exam MAS-I is a 4-hour exam with a 15-minute scheduled break (the 15 minutes will be separate from the 4-hour exam window). It will feature the following item types: (Previously, it was a purely multiple-choice exam.)

- *Multiple Choice*

  Multiple answer choices are presented after a problem with only one correct answer. Traditionally, multiple-choice questions have five answer choices, most of which are in the form of ranges, e.g.:

  A. Less than 1%
  B. At least 1%, but less than 2%
  C. At least 2%, but less than 3%
  D. At least 3%, but less than 4%
  E. At least 4%

  If your answer is much lower than the bound indicated by Answer A or much higher than that suggested by Answer E, do check your calculations. The chances are that you have made computational mistakes, but this is not definitely the case (sometimes the CAS examiners themselves made a mistake!).

- *(Partially new) Multiple Selection*

  Multiple answer choices are presented after a problem with more than one correct answer. You will have to select all choices that apply.

- *(New!) Point and Click*

  An image is presented after a problem where the candidate must identify the correct area of the image by clicking on the correct location in the image.

- *(New!) Fill in the Blank*

  A blank section is presented after a problem where the candidate must input the correct value.

The CAS has yet to release sample questions for the new item types. Regardless of the types of question, you will learn the exam material in essentially the same way, paying attention to:

- Mathematical formulas and their applications, in order to gain computational proficiency

  (This is the traditional way you approach an actuarial exam.)

- **[IMPORTANT!]** The conceptual aspects of various statistical techniques in the syllabus

  (Unlike P and FM, conceptual issues will figure prominently in MAS-I, especially in Domain C; see Chapter 11. For MAS-I, you will need to spend time understanding the nitty-gritty details of different types of predictive model.)

Historical Pass Ratios

Based on the exam statistics available from

https://www.casact.org/exams-admissions/exams-results-summary-exam-statistics,

the table below shows the number of candidates, pass ratios, and effective pass ratios for Exam MAS-I since it was offered in Spring 2018.

Sitting	# Candidates	# Passing Candidates	Pass Ratio	Effective Pass Ratio
Spring 2023	(TBA)	(TBA)	(TBA)	(TBA)
Fall 2022	845	440	52.1%	55.1%
Spring 2022	816	415	50.8%	64.8%
Fall 2021	791	437	55.2%	62.4%
Spring 2021	866	371	42.8%	50.3%
Fall 2020	1224	541	44.2%	52.3%
Spring 2020		(exam cancelled due to COVID)		
Fall 2019	992	398	40.1%	47.3%
Spring 2019	948	295	31.1%	36.4%
Fall 2018	846	259	30.6%	36.4%
Spring 2018	499	229	45.9%	50.3%

The pass ratios of MAS-I are typically in the 40-50% range, reflecting on the difficulty of this exam compared to other Associateship-level exams you have taken. It remains to be seen whether the pass ratio will go up or down after the new exam format takes effect in Fall 2023.

Exam Tables

In the exam, you will be supplied with a variety of tables, including:

- *Standard normal distribution table* (used throughout Parts I and II of this study manual)

  You will need this table for values of the standard normal distribution function or standard normal quantiles, when you work with normally distributed random variables or perform normal approximation.

- *Illustrative Life Table* (used mostly in Chapter 4 of this study manual)

  You will need this table when you are told that mortality of the underlying population follows the Illustrative Life Table.

- *A table of distributions for a number of common continuous and discrete distributions and the formulas for their moments and other probabilistic quantities* (used mostly in Part II of this study manual)

  This big table provides a great deal of information about some common (e.g., exponential, gamma, lognormal, Pareto) as well as less common distributions (e.g., inverse exponential, inverse Gaussian, Pareto, Burr, etc.). When an exam question centers on these distributions and quantities such as their means or variances are needed, consult this table.

- *Quantiles of t-distribution, F-distribution, chi-square distribution* (used in Chapters 8, 10, 12 and 13 of this study manual)

  These quantiles will be of use when you perform parametric hypothesis tests.

I strongly encourage you to download these tables from

https://www.casact.org/sites/default/files/2021-03/masi_tables.pdf

right away, print out a copy, and learn how to locate the relevant entries as you work out the examples and problems in this manual.

2 About this Study Manual

What is Special about This Study Manual?

I fully understand that you have an acutely limited amount of study time and that the MAS-I exam syllabus is insanely broad. With this in mind, the overriding objective of this study manual is to help you grasp the material in Exam MAS-I as effectively and efficiently as possible, so that you will pass the exam on your first try with ease and go on to Exam MAS-II with confidence. Here are some unique features of this manual to make this possible:

- Each chapter or section starts by explicitly stating which learning objectives and outcomes of the MAS-I exam syllabus we are going to cover, to assure you that we are on track and hitting the right target.

- The explanations in each chapter are thorough, but exam-focused and integrated with carefully chosen past exam/sample questions for illustration, so that you will learn the syllabus material effectively and efficiently. Throughout, I strive to keep you motivated by showing you how different concepts are typically tested, how different formulas are used, and where the exam focus lies in each section. As you read, you will develop a solid understanding of the concepts in MAS-I and know how to study for the exam.

- Formulas and results of utmost importance are boxed for easy identification and numbered (in the (X.X.X) format) for later references. Mnemonics and shortcuts are emphasized, so are highlights of important exam items and common mistakes committed by students.

- While the focus of this study manual is on exam preparation, I take every opportunity to explain the intuitive meaning and mathematical structure of various formulas in the syllabus. The interpretations and insights you see will foster a genuine understanding of the syllabus material and reduce the need for slavish memorization. It is my belief and personal experience that a solid understanding of the underlying concepts is always conducive to achieving good exam results.

- To succeed in any actuarial exam, I can't overemphasize the importance of practicing a wide variety of exam-type problems to sharpen your understanding and develop proficiency. This study manual embraces this learning by doing approach and intersperses its expositions with approximately **600 in-text examples** and **900 end-of-chapter/section problems** (the harder ones are labeled as [**HARDER!**] or [**VERY HARD!!**]), which are either taken/adapted from relevant SOA/CAS past exams or original, all with step-by-step solutions, to consolidate your understanding and give you a sense of what you can expect to see in the real exam. As a general guide, you should:

  - ▷ Study *all* of the in-text examples, paying particular attention to recent CAS past exam questions.
  - ▷ Work out at least *half* of the end-of-chapter/section problems. Of course, the more problems you do, the better.

- Three full-length practice exams designed to mimic the real MAS-I exam in terms of style and difficulty conclude this study manual and give you a holistic review of the syllabus material. Detailed illustrative solutions are provided.

> **NOTE**
>
> The three practice exams will be released after the CAS posts the sample questions for the new item types. They are expected to be available on www.actuarialuniversity.com in mid-September.

Contact Us

If you encounter problems with your learning, we stand ready to help.

- For **technical** issues (e.g., not able to access the manual on Actuarial University, extending your digital license, upgrading your product, exercising the Pass Guarantee), please email Customer Service at `support@actexlearning.com`.

- Questions related to **specific contents** of this manual, including potential errors (typographical or otherwise), can be directed to me (Ambrose) by emailing `ambrose-lo@uiowa.edu`.

> **NOTE**
>
> For a faster turnaround, it would be greatly appreciated if you could reach out to the appropriate email address. ☺

About the Author

Professor **Ambrose Lo**, PhD, FSA, CERA, is currently Associate Professor of Actuarial Science with tenure at the Department of Statistics and Actuarial Science, The University of Iowa. He earned his B.S. in Actuarial Science (first class honors) and PhD in Actuarial Science from The University of Hong Kong in 2010 and 2014, respectively, and attained his Fellowship of the Society of Actuaries (FSA) in 2013. He joined The University of Iowa as Assistant Professor of Actuarial Science in August 2014, and was tenured and promoted to Associate Professor in July 2019. His research interests lie in dependence structures, quantitative risk management as well as optimal (re)insurance. His research papers have been published in top-tier actuarial journals, such as *ASTIN Bulletin: The Journal of the International Actuarial Association*, *Insurance: Mathematics and Economics*, and *Scandinavian Actuarial Journal*.

Besides dedicating himself to actuarial research, Ambrose attaches equal importance to teaching and education, through which he nurtures the next generation of actuaries and serves the actuarial profession. He has taught courses on financial derivatives, mathematical finance, life contingencies, and statistics for risk modeling. He has authored or coauthored the *ACTEX Study Manuals for Exams MAS-I, MAS-II, PA, and SRM*, a *Study Manual for Exam FAM*, and the textbook *Derivative Pricing: A Problem-Based Primer* (2018) published by Chapman & Hall/CRC Press. Although helping students pass actuarial exams is an important goal of his teaching, inculcating students with a thorough understanding of the subject and concrete problem-solving skills is always his top priority. In recognition of his exemplary teaching, Ambrose has received a number of awards and honors ever since he was a graduate student, including the 2012 Excellent Teaching Assistant Award from the Faculty of Science, The University of Hong Kong, public recognition in the *Daily Iowan* as a faculty member "making a positive difference in students' lives during their time at The University of Iowa" for eight years in a row (2016 to 2023), and the 2019-2020 Collegiate Teaching Award from the College of Liberal Arts and Sciences, The University of Iowa.

Part I

Probability Models
(Stochastic Processes & Survival Models)

Chapter 2

Reliability Theory

LEARNING OBJECTIVES

5. Given the joint distribution of more than one source of failure in a system (or life) and using Poisson Process assumptions:
    - Calculate probabilities and moments associated with functions of these random variables' variances.
    - Understand difference between a series system (joint life) and parallel system (last survivor) when calculating expected time to failure or probability of failure by a certain time
    - Understand the effect of multiple sources of failure (multiple decrement) on expected system time to failure (expected lifetime)

Range of weight: 2-8 percent

Chapter overview: Reliability theory is mainly concerned with analyzing the distribution of the random lifetime of a multi-component system. In particular, we are interested in the probability that such a system will function at a certain point of time. This probability is termed the reliability of the system. In the first two sections of this relatively short chapter, we look at a system from a static point of view, focusing on whether the system operates at a particular point of time from the knowledge of which components are operating and on computing the reliability of the system. Several popular designs to arrange the components of a system are presented, and the resulting influence on the reliability of a system is examined. In Section 2.3, we prescribe probability distributions on the individual components of a system and investigate the overall lifetime distribution of the system dynamically over time.

2.1 Typical Systems

MAS-I KNOWLEDGE STATEMENT(S)

c. Time until failure of the system (life)

d. Time until failure of the system (life) from a specific cause

e. Time until failure of the system (life) for parallel or series systems with multiple components

f. Paths that lead to parallel or series system failure for systems with multiple components

g. Relationship between failure time and minimal path and minimal cut sets

h. Bridge system and defining path to failure

OPTIONAL SYLLABUS READING(S)

Ross, Sections 9.1 and 9.2

We start by introducing several common system designs, which permeate much of the whole chapter.

Series systems. A *series system* functions if and only if *all* of its components function. It fails as soon as one of the components fails. To express this fact mathematically, we define the binary[i] variable

$$x_i = \begin{cases} 1, & \text{if the } i\text{th component is functioning,} \\ 0, & \text{if the } i\text{th component has failed,} \end{cases}$$

to be the *state* of the i^{th} component and designate the vector $\mathbf{x} = (x_1, \ldots, x_n)$ representing which components are functioning as the *state vector* of the system. Corresponding to a state vector $\mathbf{x}$, the *structure function* of the system defined as

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if the system is functiong when the state vector is } \mathbf{x}, \\ 0, & \text{if the system has failed when the state vector is } \mathbf{x}, \end{cases}$$

indicates whether the system as a whole is functioning. It is a convenient device which characterizes a system in the sense that different systems are distinguished by different structure functions ϕ.

For a series system, the structure function is given by

$$\phi(\mathbf{x}) = \prod_{i=1}^{n} x_i = \begin{cases} 1, & \text{if every component is functioning, i.e., } x_i = 1 \text{ for all } i, \\ 0, & \text{if any component has failed, i.e., } x_i = 0 \text{ for some } i. \end{cases}$$

It may help to understand the meaning of a series system by representing it pictorially in Figure 2.1.1, where a signal initiated at the left must pass through each and every component for reception on the right (i.e., for the system to work).

[i] A binary variable takes the values of 0 and 1 only.

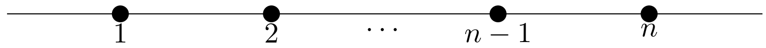


Figure 2.1.1: Pictorial representation of an n-component series system.

Parallel systems. In contrast to a series system, a *parallel system* functions if and only if *at least one* of its components is functioning. The corresponding structure function is

$$\begin{aligned}\phi(\mathbf{x}) &= \boxed{\max(x_1, \ldots, x_n)} \\ &= \begin{cases} 1, & \text{if at least one component is functioning, i.e., } x_i = 1 \text{ for some } i, \\ 0, & \text{if all components have failed, i.e., } x_i = 0 \text{ for all } i. \end{cases}\end{aligned}$$

A pictorial illustration of a parallel system is given in Figure 2.1.2.

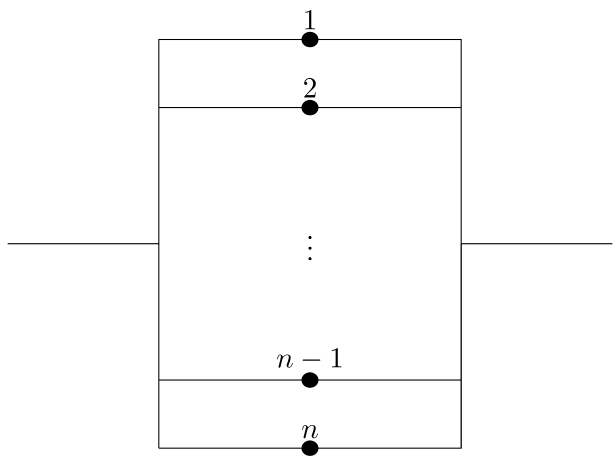


Figure 2.1.2: Pictorial representation of an n-component parallel system.

Other systems. Series and parallel systems form the basic building blocks of more complex systems. Here are some examples:

1. *k-out-of-n systems* (Example 9.3 of Ross): A *k-out-of-n system* with $1 \le k \le n$ is a generalization of series and parallel systems. It functions if and only if at least k out of the n components are functioning. The corresponding structure function is

$$\phi(\mathbf{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n x_i \ge k, \\ 0, & \text{if } \sum_{i=1}^n x_i < k. \end{cases}$$

   In particular, an n-out-of-n system is a series system (the functioning of all of the n components is required) and a 1-out-of-n system becomes a parallel system (as long as one component works, so does the whole system).

2. *A four-component hybrid system* (Example 9.4 of Ross): A system can consist of components which are themselves series, parallel, or k-out-of-n systems. As an illustration, consider the four-component system described in Figure 2.1.3. Observe that components 3 and 4 constitute a parallel system, which, together with components 1 and 2, forms a

series system—the system works if and only if both component 1 and component 2 ("series"), and at least one of components 3 and 4 work ("parallel"). The structure function of this hybrid system is given by

$$\phi(\mathbf{x}) = \boxed{\overbrace{x_1 x_2 \times \underbrace{\max(x_3, x_4)}_{\text{parallel}}}^{\text{series}}},$$

which, along with the use of the useful identity

$$\boxed{\max(x_1, \ldots, x_n) = 1 - \prod_{i=1}^{n}(1 - x_i)} \quad (2.1.1)$$

for binary variables $x_1, \ldots, x_n$, can be further written as

$$\phi(\mathbf{x}) = x_1 x_2[1 - (1 - x_3)(1 - x_4)] = \boxed{x_1 x_2(x_3 + x_4 - x_3 x_4)}.$$

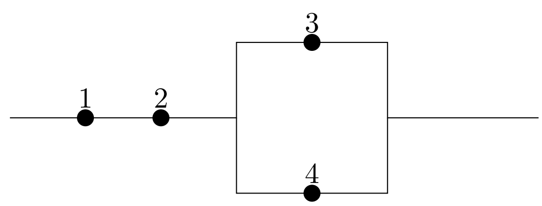


Figure 2.1.3: Pictorial representation of a four-component hybrid system.

3. *Bridge system* (Example 9.8 of Ross): A special type of systems explicitly stated in the knowledge statements of the exam syllabus is known as a bridge system, which is depicted in Figure 2.1.4. You may think of the path passing through component 3 in the middle as a "bridge."

   Expressing a bridge in terms of series and parallel structures by inspection is not an easy task. This will be achieved with the aid of minimal path and cut sets to be introduced in the next paragraph.

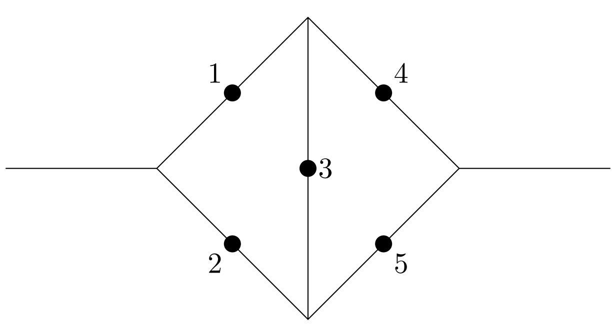


Figure 2.1.4: Pictorial representation of a bridge system.

[HARDER!] Minimal path and minimal cut sets. A deeper understanding of a system can be acquired by studying which critical components are sufficient for the system to work, and failure of which components alone will jeopardize the functioning of the system. These are formalized by the concepts of minimal path and minimal cut sets, which enable us to express an arbitrary system as a parallel representation of series structures, and as a series representation of parallel structures.

- *Minimal path sets:* A *minimal*[ii] *path set* is a set of components satisfying both properties below:

  1. *("Path")* The functioning of *each and every* component in this set guarantees the functioning of the whole system.
  2. *("Minimal")* The same conclusion in Point 1 cannot be drawn if you take away any component from the set.

  The above definition of a minimal path set is a bit delicate. Note that:

  ▷ Minimal path set is not unique. For an n-component parallel system, the sets $\{1\}$, $\{2\}$, ..., $\{n-1\}$, $\{n\}$ are all minimal path sets; each component alone constitutes a minimal path set. More generally,

  for a k-out-of-n system, there are $\dbinom{n}{k}$ minimal path sets,

  namely, all of the sets comprising exactly k components; this is the content of Example 9.6 of Ross. This is because any k components suffice to ensure that the whole system functions and there are $\binom{n}{k}$ such combinations. Setting $k = 1$ and $k = n$, we get $\binom{n}{1} = n$ minimal path sets for an n-component parallel system and $\binom{n}{n} = 1$ minimal path set for an n-component series system.

  ▷ Different minimal path sets may also have different number of components. To see this, consider the bridge system depicted in Figure 2.1.4, for which the minimal path sets include $\{1,4\}, \{1,3,5\}, \{2,5\}$, and $\{2,3,4\}$. Some minimal path sets have two components while some have three.

  ▷ The fact that the system functions does not require each component in *all* minimal path sets to work. To put it in another way, the fact that a component in the set fails does *not* mean that the whole system fails! In the bridge system example, it suffices for the system to function with components 1 to 4 operating only, but with components 2 and 5 both failed (although $\{2,5\}$ is also a minimal path set).

  By the definition of minimal path sets, a system will function if and only if there is *at least one* minimal path set in which *all* components work. Notice the important order of the phrases "at least one" and "all". Recalling that "at least one" suggests a parallel structure while "all" means a series structure, we manage to express the structure function of any

[ii] In mathematics, there is a distinction between "minimal" and "minimum". We shall not pursue the subtle but substantive difference here.

system as a parallel representation of series systems:

$$\phi(\mathbf{x}) = \underbrace{\max_j}_{\text{parallel}} \underbrace{\prod_{i \in A_j}}_{\text{series}} x_i, \tag{2.1.2}$$

where the A_j's are the minimal path sets of the system. To check that (2.1.2) is true, we note the following equivalences:

$$\begin{aligned} \phi(\mathbf{x}) = 1 \quad &\Leftrightarrow \quad \text{there exists a minimal path set } A_j \text{ such that } \prod_{i \in A_j} x_i = 1 \\ &\Leftrightarrow \quad \text{there exists a minimal path set } A_j \text{ such that } x_i = 1 \text{ for all } i \in A_j \end{aligned}$$

The last statement is simply the definition of a minimal path set.

Example 2.1.1. **(Based on Example 9.5 of Ross: Identification of minimal path set for a hybrid system)** Consider the five-component system in Figure 2.1.5.

Determine the number of minimal path sets of the system.

A. 1

B. 2

C. 3

D. 4

E. 5

Solution. By inspection, there are four minimal path sets: $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 5\}$ and $\{2, 5\}$. **(Answer: D)** □

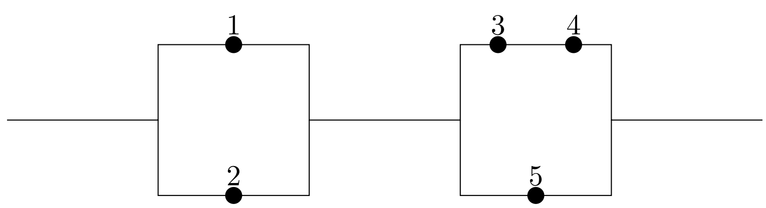


Figure 2.1.5: Pictorial representation of the five-component system in Example 2.1.1.

The next two Exam MAS-I/S past exam problems both center on the minimal path sets of k-out-of-n systems (series and parallel systems, in particular).

Example 2.1.2. **(CAS Exam S Fall 2017 Question 8: Two k-out-of-n systems connected in parallel)** A 3-out-of-50 system is placed in parallel with a 48-out-of-50-system to form a combined system.

Calculate the number of minimal path sets for the combined system.

A. Fewer than 20,000

B. At least 20,000, but fewer than 30,000

C. At least 30,000, but fewer than 40,000

D. At least 40,000, but fewer than 50,000

E. At least 50,000

Solution. The 3-out-of-50 system and 48-out-of-50 have $\binom{50}{3} = 19,600$ and $\binom{50}{48} = 1,225$ minimal path sets, respectively. When the two systems are put in parallel, each minimal path set of any of the two systems provides a minimal path set of the aggregate system. The number of minimal path sets increases to $19,600 + 1,225 = \boxed{20,825}$. **(Answer: B)** □

Example 2.1.3. **(CAS Exam MAS-I Fall 2018 Question 7: Two k-out-of-n systems connected in series)** A 3-out-of-50 system is placed in series with a 48-out-of-50-system.

Calculate the number of minimal path sets.

A. Fewer than 20,000

B. At least 20,000 but fewer than 100,000

C. At least 200,000 but fewer than 2,000,000

D. At least 2,000,000 but fewer than 20,000,000

E. At least 20,000,000

Ambrose's comments: This MAS-I problem is an adaptation of the preceding Exam S problem, with "placed in parallel" replaced by "placed in series."

Solution. Following the solution of the preceding example, the 3-out-of-50 system and 48-out-of-50-system have, respectively, $\binom{50}{3} = 19,600$ and $\binom{50}{48} = 1,225$ minimal path sets. When the two systems are put in series, a minimal path set of the aggregate system comprises a minimal path set from the 3-out-of-50 system and another minimal path set from the 48-out-of-50-system. Overall, the number of minimal path sets is $19,600 \times 1,225 = \boxed{24,010,000}$. **(Answer: E)** □

Remark. Answer B is probably intended to be "At least 20,000 but fewer than <u>2</u>00,000".

- *Minimal cut sets:* As a concept dual to a minimal path set, a *minimal cut set* consists of a set of components such that:

  1. *("Cut")* The failure of *all* components in the set guarantees the failure of the whole system.
  2. *("Minimal")* The same conclusion cannot be drawn if you take away any component from the set.

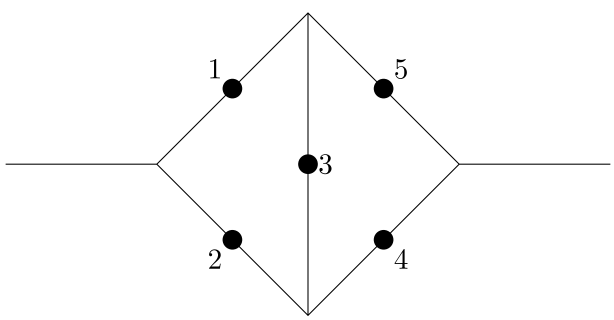
  By the definition of a minimal cut set, a system will fail if and only if there is *at least one* minimal cut set in which *all* components fail. This translates into the following *series* arrangement of parallel systems:

$$\boxed{\phi(\mathbf{x}) = \underbrace{\prod_j}_{\text{series}} \underbrace{\max_{i \in C_j}}_{\text{parallel}} x_i,} \tag{2.1.3}$$

  where the C_j's are the minimal cut sets. As a check, note the following equivalences:

$$\begin{aligned} \phi(\mathbf{x}) = 0 \quad &\Leftrightarrow \quad \text{there exists a minimal cut set } C_j \text{ such that } \max_{i \in C_j} x_i = 0 \\ &\Leftrightarrow \quad \text{there exists a minimal cut set } C_j \text{ such that } x_i = 0 \text{ for all } i \in C_j \end{aligned}$$

Example 2.1.4. (CAS Exam MAS-I Spring 2018 Question 9: Minimal path and cut sets of a bridge system) You are given the following system:



and the following statements:

I. $\{1, 5\}$ and $\{2, 4\}$ are minimal path sets.

II. $\{1, 2\}$ and $\{4, 5\}$ are minimal cut sets.

III. $\{1, 3, 5\}$ and $\{2, 3, 4\}$ are both minimal path sets and minimal cut sets.

Determine which of the above statements are correct.

A. None are correct

B. I and II only

C. I and III only

D. II and III only

E. The answer is not given by (A), (B), (C), or (D)

Solution. I&II. Correct, by inspection.

III. Incorrect. Because $\{1,5\}$ is a minimal path set, $\{1,3,5\}$, with component 3 added, cannot serve as a minimal path set. The same idea applies to $\{2,3,4\}$. **(Answer: B)**

□

Remark. Statement III would be true if it says "$\{1,3,4\}$ and $\{2,3,5\}$ are both minimal path sets and minimal cut sets."

Example 2.1.5. (Based on Example 9.8 of Ross: Manipulating the structure function of a bridge system) Determine which of the following representation of the structure function of the bridge system in Figure 2.1.4 is correct.

A. $x_3 \max(x_1, x_2) \max(x_4, x_5)$

B. $x_3 \max(x_1 x_4, x_2 x_5)$

C. $(1 - x_3) \max(x_1, x_4) \max(x_2, x_5)$

D. $(1 - x_1 x_4)(1 - x_1 x_3 x_5)(1 - x_2 x_5)(1 - x_2 x_3 x_4)$

E. $1 - (1 - x_1 x_4)(1 - x_1 x_3 x_5)(1 - x_2 x_5)(1 - x_2 x_3 x_4)$

Solution. Since the minimal path sets of the bridge system are $\{1,4\}, \{1,3,5\}, \{2,5\}$ and $\{2,3,4\}$, using the parallel representation of series systems given in (2.1.2) we have

$$\max(x_1 x_4, x_1 x_3 x_5, x_2 x_5, x_2 x_3 x_4) = 1 - (1 - x_1 x_4)(1 - x_1 x_3 x_5)(1 - x_2 x_5)(1 - x_2 x_3 x_4).$$

Thus **Answer E** is correct. □

Remark. As the minimal cut sets of the bridge system are $\{1,2\}, \{1,3,5\}, \{2,3,4\}$ and $\{4,5\}$, we can also use (2.1.3) to express the structure function as

$$\begin{aligned}
& \max(x_1, x_2) \max(x_1, x_3, x_5) \max(x_2, x_3, x_4) \max(x_4, x_5) \\
= \; & [1 - (1 - x_1)(1 - x_2)][1 - (1 - x_1)(1 - x_3)(1 - x_5)] \\
& \quad \times [1 - (1 - x_2)(1 - x_3)(1 - x_4)][1 - (1 - x_4)(1 - x_5)] \\
= \; & (x_1 + x_2 - x_1 x_2)(x_4 + x_5 - x_4 x_5) \\
& \quad \times [1 - (1 - x_1)(1 - x_3)(1 - x_5)][1 - (1 - x_2)(1 - x_3)(1 - x_4)].
\end{aligned}$$

This does not look like the structure function based on the parallel representation of series systems, but it can be shown that they are algebraically identical to each other.

Example 2.1.6. **[HARDER!] (Given minimal path sets, find minimal cut sets)** You are given that the minimal path sets of a system are $\{1,5\}, \{2,5\}, \{1,3,4\}$, and $\{2,3,4\}$.

Calculate the number of minimal cut sets of the system.

A. 1

B. 2

C. 3

D. 4

E. 5

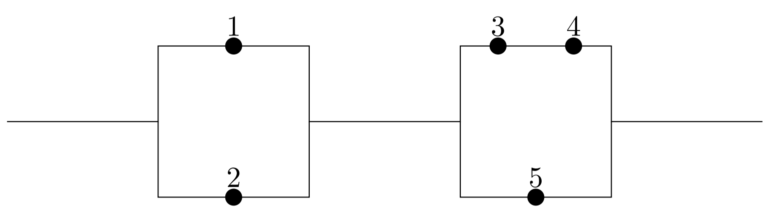
Solution 1. The key to solving this example is to realize that a cut set must have *at least one element* of every minimal path set. If this is not the case, then the functioning of all components in the excluded minimal path set will guarantee the functioning of the whole system, even if all components in a minimal cut set fail. This results in a contradiction.

Choosing one element from each of the four minimal path sets and ensuring minimality (this requires experimentation), we get three minimal cut sets:

$$\begin{array}{llllll} \{\boxed{1}, 5\}, & \{\boxed{2}, 5\}, & \{\boxed{1}, 3, 4\}, & \{\boxed{2}, 3, 4\} & \to & \{1,2\} \\ \{1, \boxed{5}\}, & \{2, \boxed{5}\}, & \{1, \boxed{3}, 4\}, & \{2, \boxed{3}, 4\} & \to & \{3,5\} \quad \textbf{(Answer: C)} \\ \{1, \boxed{5}\}, & \{2, \boxed{5}\}, & \{1, 3, \boxed{4}\}, & \{2, 3, \boxed{4}\} & \to & \{4,5\} \end{array}$$

□

Solution 2 (Minimal path sets → system → minimal cut sets). With some experimentation, we can do some reverse engineering and construct the system from the given minimal path sets:



Visually inspecting the system, we find that there are three minimal cut sets, $\{1,2\}, \{3,5,\}, \{4,5\}$. **(Answer: C)** □

Remark. In this example, the system is simple enough so that reverse engineering used in Solution 2 is possible. For slightly more complex systems, reverse engineering will be very difficult, if not impossible (see Problems 2.4.6 and 2.4.7, which are exercises from Ross). That is why the technique illustrated in Solution 1 is still valuable.

2.2 Reliability of Systems of Independent Components

MAS-I KNOWLEDGE STATEMENT(S)

h. Bridge system and defining path to failure

i. Random graphs and defining path to failure

l. Method of inclusion and exclusion as applied to failure time estimates

OPTIONAL SYLLABUS READING(S)

Ross, Sections 9.3 and 9.4

Reliability. Probability enters our conversations in this section, where we would like to determine the probability that the overall system is functioning at a given time point of interest. Assuming that the states of the components $\mathbf{X} = (X_1, \ldots, X_n)$ (Note: We switch notation from x_i to capital letter X_i because the states are now random variables) are independent, we aim to compute the *reliability* (or reliability function) of the system defined by

$$\boxed{r(\mathbf{p}) = \mathrm{E}[\phi(\mathbf{X})] = \Pr(\phi(\mathbf{X}) = 1)},$$

which is a function of $\mathbf{p} = (p_1, \ldots, p_n)$, the vector of component reliabilities. In short, the reliability of a system is simply the probability that it will function at a reference time point.

As the following systems show, the ability to write the structure function of a system proficiently is the key to success when it comes to calculating reliability.

- *Series systems:* The reliability of a series system is

$$r(\mathbf{p}) = \Pr(X_i = 1 \text{ for all } i = 1, \ldots, n) = \prod_{i=1}^{n} \Pr(X_i = 1) = \boxed{\prod_{i=1}^{n} p_i}, \tag{2.2.1}$$

  in which the independence of the states of the components is used in the second equality. This expression is also the same as the structure function of the series system with all x_i's replaced by p_i's upon taking expectation:

$$r(\mathbf{p}) = \mathrm{E}[X_1 X_2 \cdots X_n] \overset{\text{(independence)}}{=} \mathrm{E}[X_1]\mathrm{E}[X_2] \cdots \mathrm{E}[X_n] = \prod_{i=1}^{n} p_i.$$

- *Parallel systems:* The reliability of a parallel system is

$$r(\mathbf{p}) = \Pr(X_i = 1 \text{ for some } i) = 1 - \Pr(X_i = 0 \text{ for all } i) = \boxed{1 - \prod_{i=1}^{n}(1 - p_i)}. \tag{2.2.2}$$

When $n = 2$, the preceding expression simplifies to[iii]

$$\boxed{p_1 + p_2 - p_1p_2,} \tag{2.2.3}$$

which is a simple result you may consider remembering.

Example 2.2.1. (CAS Exam S Fall 2016 Question 8: Reliability of a parallel system with uniform components) For a parallel system with two independent machines, you are given the following information:

- The hazard rate for each machine is:

$$\mu_x = \frac{1}{100 - x}, \quad \text{for } 0 \le x < 100; \ x \text{ in months}$$

- One machine has worked for 40 months and the other machine has worked for 60 months.

Calculate the probability that the system will function for 20 more months.

A. Less than 0.35

B. At least 0.35, but less than 0.55

C. At least 0.55, but less than 0.75

D. At least 0.75, but less than 0.95

E. At least 0.95

Solution. From the given hazard rate (or failure rate), we can determine the survival function of a *newborn* machine:

$$S(x) = \exp\left(-\int_0^x \frac{1}{100 - t}\,\mathrm{d}t\right) = \exp[\ln(100 - t)|_0^x] = 1 - \frac{x}{100}$$

for $0 \le x \le 100$. The probability that the machine aged 40 months will function for 20 more months is

$$p_1 = \frac{S(\overbrace{60}^{40+20})}{\underbrace{S(40)}_{\substack{\text{conditional on living}\\ \text{beyond 40 months}}}} = \frac{1 - 60/100}{1 - 40/100} = \frac{2}{3}$$

and the probability that the machine aged 60 months will function for 20 more months is

$$p_2 = \frac{S(80)}{S(60)} = \frac{1 - 80/100}{1 - 60/100} = \frac{1}{2}.$$

[iii]Students with more training in set theory may associate $p_1 + p_2 - p_1p_2$, with the aid of a Venn diagram, with the probability of the union of the set where Component 1 works and the set where Component 2 works. The subtraction by p_1p_2 serves to avoid double counting.

By (2.2.3), the reliability is

$$p_1 + p_2 - p_1 p_2 = \frac{2}{3} + \frac{1}{2} - \frac{2}{3}\left(\frac{1}{2}\right) = \boxed{\frac{5}{6} = 0.8333}. \quad \textbf{(Answer: D)}$$

□

Remark. (i) In fact, the remaining lifetime of a machine aged x months is uniformly distributed over $[0, 100 - x]$.

(ii) Suppose that the last line of the question is changed to "Calculate the probability that the system will function for *less than* 20 months." With $q_1 = 1 - p_1 = 1/3$, $q_2 = 1 - p_2 = 1/2$, you may be tempted to calculate the probability as

$$q_1 + q_2 - q_1 q_2 = \frac{2}{3},$$

which is not the same as one minus the answer in the original version of the example, namely $1 - 5/6 = 1/6$. In other words, we cannot simply replace the p's by the q's to find the probability that the system will fail. In fact, in terms of the q's the probability that the system will function for less than 20 months is $q_1 q_2 = 1/6$.

Example 2.2.2. **[HARDER!] (CAS Exam MAS-I Fall 2019 Question 7: Reliability of a series system with exponential and uniform components)** You are given the following information regarding a series system with two independent machines, X and Y:

- The hazard rate function, in years, for machine i is denoted by $r_i(t)$
- $r_X(t) = \ln(1.06)$, for $x > 0$
- $r_Y(t) = \frac{1}{20-t}$, for $0 < y < 20$
- Both machines are currently three years old

Calculate the probability that the system fails when the machines are between five and nine years old.

A. Less than 0.305

B. At least 0.305, but less than 0.315

C. At least 0.315, but less than 0.325

D. At least 0.325, but less than 0.335

E. At least 0.335

Ambrose's comments: In the second and third points of this MAS-I exam question, "$x > 0$" and "$0 < y < 20$" should be "$t > 0$" and "$0 < t < 20$," respectively.

Solution. We are asked to find the reliability of the series system for five years old less the reliability for nine years old. By (1.2.3),

$$S_X(t) = e^{-(\ln 1.06)t} \text{ for } t > 0, \qquad \text{and} \qquad S_Y(t) = 1 - \frac{t}{20} \text{ for } 0 < t < 20.$$

Given that the system is currently three years old, the (conditional) probability that the series system is still functioning at five years old is

$$\underbrace{\frac{S_X(5)}{S_X(3)} \times \frac{S_Y(5)}{S_Y(3)}}_{(2.2.1)} = e^{-(\ln 1.06)2} \times \frac{1 - 5/20}{1 - 3/20} = 0.785291$$

and the probability that it is still functioning at nine years old is

$$\frac{S_X(9)}{S_X(3)} \times \frac{S_Y(9)}{S_Y(3)} = e^{-(\ln 1.06)6} \times \frac{1 - 9/20}{1 - 3/20} = 0.456151.$$

The required probability is $0.785291 - 0.456151 = \boxed{0.3291}$. **(Answer: D)** □

Example 2.2.3. **[HARDER!] (CAS Exam MAS-I Spring 2019 Question 7: Constructing a parallel system with non-identically distributed components)** You are given the following information:

- In a toolbox there are two types of components that all perform the same function:
  - ▷ There are 4 components of type A, each with reliability of 0.600
  - ▷ There are 20 components of type B, each with reliability of 0.300
  - ▷ All components are independent
- Using only the components in this toolbox, you want to construct a parallel system with a reliability of at least 0.995

Calculate the minimum number of components needed to create this system.

A. Fewer than 3

B. At least 3, but fewer than 5

C. At least 5, but fewer than 7

D. At least 7, but fewer than 9

E. At least 9

Solution. Let n be the number of components needed.

Intuitively, it makes sense to first use components of type A, which have a higher reliability, then use components of type B if needed. If only the 4 components of type A are used to construct a parallel system, the resulting reliability, by (2.2.2), is $1-(1-0.6)^4 = 0.9774$, which is lower than 0.995 and so components of type B are also needed. For the reliability of the system to be at least 0.995, we need $1-(1-0.6)^4(1-0.3)^{n-4} > 0.995$, which gives

$$0.7^{n-4} < \frac{0.005}{0.0256} \quad \Rightarrow \quad n > 4 + \frac{\ln(0.005/0.0256)}{\ln 0.7} = 8.5788.$$

The least integral value of n is $\boxed{9}$. **(Answer: E)** □

Remark. (i) Intuition suggests and (2.2.1) and (2.2.2) both confirm that adding components to a parallel system increases its reliability, whereas adding components to a series system decreases its reliability.

(ii) If you overlook the fact that there are only 4 components of type A and mistakenly solve the inequality $1-(1-0.6)^n > 0.995$, you would get $n > 5.7824$, leading to Answer C.

- *k-out-of-n systems:* The reliability function of a general k-out-of-n system is the probability that at least k out of the n components work. In the case of i.i.d. components with $p := p_1 = \cdots = p_n$,[iv] this is

$$r(\mathbf{p}) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

For non-i.i.d. components, it is not easy to display the reliability explicitly, except when k and n are small. For $k = 2$ and $n = 3$ (i.e., a 2-out-of-3 system), the reliability function is

$$\begin{aligned} r(\mathbf{p}) &= \Pr\left(X = (1,1,1),(1,1,0),(1,0,1) \text{ or } (0,1,1)\right) \\ &= \Pr(\text{exactly 2 components work}) + \Pr(\text{exactly 3 components work}) \\ &= p_1p_2(1-p_3) + p_1p_3(1-p_2) + p_2p_3(1-p_1) + p_1p_2p_3 \\ &= \boxed{p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3}, \end{aligned} \tag{2.2.4}$$

which simplifies, in the case of i.i.d. components, to

$$r(\mathbf{p}) \overset{\text{i.i.d.}}{=} \boxed{3p^2 - 2p^3}. \tag{2.2.5}$$

Question: What is the reliability function of a general 3-out-of-4 system? (this is Example 9.14 of Ross)

Answer: $p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4 - 3p_1p_2p_3p_4 \overset{\text{i.i.d.}}{=} 4p^3 - 3p^4$.

[iv] Throughout this study manual, the symbol ":=" means "is defined as."

Example 2.2.4. **(CAS Exam MAS-I Spring 2018 Question 8: 3-out-of-5 system)** You are given a system of five independent components, with each component having reliability of 0.90. Three-out-of-five of the components are required to function for the system to function.

Calculate the reliability of this three-out-of-five system.

A. Less than 0.96

B. At least 0.96, but less than 0.97

C. At least 0.97, but less than 0.98

D. At least 0.98, but less than 0.99

E. At least 0.99

Solution. The reliability of a 3-out-of-5 system in the i.id. case is

$$\begin{aligned} r(p) &= \sum_{i=3}^{5} \binom{5}{i} p^i (1-p)^{5-i} \\ &= 10p^3(1-p)^2 + 5p^4(1-p) + p^5, \end{aligned}$$

which, when $p = 0.9$, equals $\boxed{0.99144}$. **(Answer: E)** □

Let's work out some examples involving more sophisticated systems.

Example 2.2.5. **[HARDER!] (CAS Exam S Fall 2016 Question 9: Given minimal path sets, find the reliability)** You are given the following information:

- A system has two minimal path sets: $\{1, 2, 4\}$ and $\{1, 3, 4\}$.
- Reliability for components 1 and 2 is uniformly distributed from 0 to 1.
- Reliability for components 3 and 4 is uniformly distributed from 0 to 2.
- All components in the system are independent.
- You are starting at time 0.

Calculate the probability that the lifetime of the system will be less than 0.25.

A. Less than 0.30

B. At least 0.30, but less than 0.35

C. At least 0.35, but less than 0.40

D. At least 0.40, but less than 0.45

E. At least 0.45

Ambrose's comments: This past exam problem challenges you by not directly telling you the design of the system. The knowledge of minimal path sets is required.

Solution. Using the parallel representation of series systems in Section 2.1, we can express the structure function of the given system algebraically as

$$\begin{aligned}\phi(\mathbf{x}) &= \max(x_1x_2x_4, x_1x_3x_4) \\ &\overset{(2.1.1)}{=} x_1x_2x_4 + x_1x_3x_4 - (x_1x_2x_4)(x_1x_3x_4) \\ &\overset{(x_1^2=x_1, x_4^2=x_4)}{=} x_1x_2x_4 + x_1x_3x_4 - x_1x_2x_3x_4 \\ &= x_1(x_2+x_3-x_2x_3)x_4.\end{aligned}$$

Alternatively, one may observe from the two minimal path sets that the system functions if and only if both components 1 and 4 function (series), and at least one of components 2 and 3 function (parallel). This observation also leads to the above structure function.

Now replacing x_1 and x_2 by $p_1 = p_2 = 1 - 0.25/1 = 0.75$ and x_3 and x_4 by $p_3 = p_4 = 1 - 0.25/2 = 0.875$, we can calculate the probability (i.e., reliability) that the system function for more than 0.25 units as

$$p_1(p_2 + p_3 - p_2p_3)p_4 = 0.75(0.75 + 0.875 - 0.75 \times 0.875)(0.875) = 0.6357.$$

Finally, the probability that the lifetime of the system will be less than 0.25 is $1 - 0.6357 = \boxed{0.3643}$. **(Answer: C)** □

Remark. (i) The four-component system is taken from Example 9.16 on page 567 of Ross. In fact, it is the same in structure as the four-component system in Figure 2.1.3 – just move component 2 therein to the right and relabel the four components.

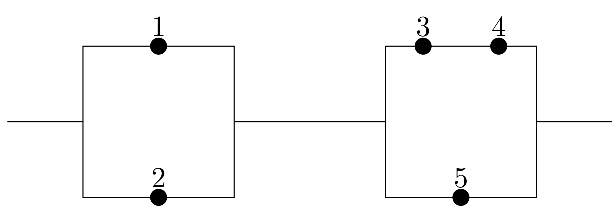
(ii) The minimal cut sets of the system are $\{1\}, \{2, 3\}$ and $\{4\}$. Giving you these three minimal cut sets immediately leads to $\phi(\mathbf{x}) = x_1(x_2 + x_3 - x_2x_3)x_4$ and may make the question too simple.

(iii) If you calculate the required probability (incorrectly) as

$$0.25(0.25 + 0.125 - 0.25 \times 0.125)(0.125) = 0.0107,$$

make sure that you read Remark (ii) of Example 2.2.1.

Example 2.2.6. (Reliability calculations for a hybrid system) Consider the following five-component system:



Each component functions independently with probability 0.8.

Calculate the reliability of the system.

A. Less than 0.6

B. At least 0.6, but less than 0.7

C. At least 0.7, but less than 0.8

D. At least 0.8, but less than 0.9

E. At least 0.9

Solution. This is a hybrid system consisting of a series structure and two "big" components:

1. A parallel structure with two components: Component 1 and component 2
2. A parallel structure with two components: The series structure of $\{3,4\}$ and component 5

The structure function of the system is

$$\phi(\mathbf{x}) = \underbrace{\max(x_1, x_2)}_{\text{parallel}} \overbrace{\times}^{\text{series}} \underbrace{\max(x_3x_4, x_5)}_{\text{parallel}} \overset{(2.1.1)}{=} (x_1 + x_2 - x_1x_2)(x_3x_4 + x_5 - x_3x_4x_5).$$

Replacing the x_i's by the Bernoulli random variables X_i's and taking expectations, we have

$$r(p) = (2p - p^2)(p^2 + p - p^3) \overset{(p=0.8)}{=} \boxed{0.89088}. \quad \textbf{(Answer: D)}$$

□

Remark. (If you are interested...) You may wonder: Can this example be done by identifying the structure function of the hybrid system using the parallel representations of series systems, (2.1.2), and taking expectation of the random states X_i's, as in the preceding example? The answer is affirmative, but the solution will become much more tedious.

The minimal paths of the hybrid system are $\{1,5\}, \{2,5\}, \{1,3,4\}, \{2,3,4\}$. By (2.1.2), the structure function is

$$\begin{aligned} \phi(\mathbf{x}) &= \max(x_1x_5, x_2x_5, x_1x_3x_4, x_2x_3x_4) \\ &\overset{(2.1.1)}{=} 1 - (1 - x_1x_5)(1 - x_2x_5)(1 - x_1x_3x_4)(1 - x_2x_3x_4). \end{aligned}$$

You may be tempted to take expectation and convert all of the x_i's in this expression to $p = 0.8$. This is not correct because although the X_i's are mutually independent, the four terms in the product $(1 - X_1X_5)(1 - X_2X_5)(1 - X_1X_3X_4)(1 - X_2X_3X_4)$ involve duplicated X_i's. For example, X_5 appears in both the first and second terms. Without independence, the expectation of a product is no longer the product of expectations, i.e., we generally have

$$\mathrm{E}[(1 - X_1X_5)(1 - X_2X_5)(1 - X_1X_3X_4)(1 - X_2X_3X_4)]$$

$$\begin{aligned}&\neq \mathrm{E}[(1-X_1X_5)]\mathrm{E}[(1-X_2X_5)]\mathrm{E}[(1-X_1X_3X_4)]\mathrm{E}[(1-X_2X_3X_4)]\\&= (1-p^2)^2(1-p^3)^2.\end{aligned}$$

For a correct solution, we have no choice but to expand the product and simplify the structure function, quite laboriously, as

$$\begin{aligned}\phi(\mathbf{x}) &= 1-(1-x_1x_5)(1-x_2x_5)(1-x_1x_3x_4)(1-x_2x_3x_4)\\&\overset{(x_i^2=1)}{=} 1-(1-x_1x_5-x_2x_5+x_1x_2x_5)(1-x_1x_3x_4-x_2x_3x_4+x_1x_2x_3x_4)\\&= 1-[(1-x_1x_3x_4-x_2x_3x_4+x_1x_2x_3x_4)\\&\quad+(-x_1x_5+x_1x_3x_4x_5+\underbrace{x_1x_2x_3x_4x_5-x_1x_2x_3x_4x_5}_{\text{cancel}})\\&\quad+(-x_2x_5+x_1x_2x_3x_4x_5+x_2x_3x_4x_5-x_1x_2x_3x_4x_5)\\&\quad+(x_1x_2x_5-x_1x_2x_3x_4x_5\underbrace{-x_1x_2x_3x_4x_5+x_1x_2x_3x_4x_5}_{\text{cancel}})]\\&= 1-[(1-x_1x_3x_4-x_2x_3x_4+x_1x_2x_3x_4)+(-x_1x_5+x_1x_3x_4x_5)\\&\quad+(-x_2x_5+x_2x_3x_4x_5)+(x_1x_2x_5-x_1x_2x_3x_4x_5)].\end{aligned}$$

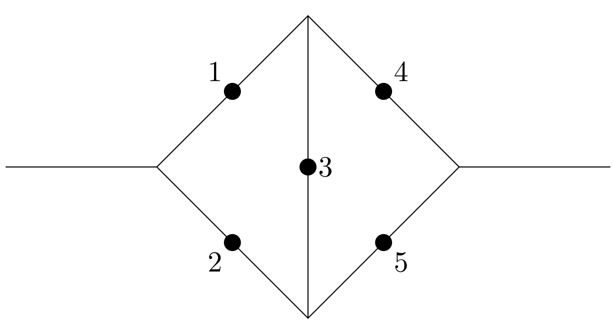
Note that each term in the preceding expression involves the x_i's of distinct components. Replacing the x_i's by the random variables X_i's and taking expectation, we have

$$\begin{aligned}r(p) &= 1-[(1-2p^3+p^4)+(-p^2+p^4)+(-p^2+p^4)+(p^3-p^5)]\\&= 2p^2+p^3-3p^4+p^5\\&\overset{(p=0.8)}{=} \boxed{0.89088}.\end{aligned}$$

(Answer: D)

The message here is that the use of (2.1.2) or (2.1.3) to construct the structure function of a system is correct, but often not the most efficient method, especially when the number of minimal path or cut sets is larger than three. Visual inspection is often what you will use on an exam!

Example 2.2.7. [HARDER!] (Based on Exercise 9.14 of Ross: Reliability of a bridge system) Consider the following bridge system in which all components function independently with probability 0.7.



Calculate the reliability of the bridge system.

A. Less than 0.60

B. At least 0.60, but less than 0.70

C. At least 0.70, but less than 0.80

D. At least 0.80, but less than 0.90

E. At least 0.90

Solution. Before calculating the numerical answer for this question, for edification we pursue higher generality and assume that component i functions with probability p_i, for $i = 1, 2, \ldots, 5$. The trick to evaluate the reliability of a bridge system easily is to condition on whether component 3 (the "bridge") is working.

Case 1. If component 3 is working, then the whole system functions if and only if both of (X_1, X_2) and (X_4, X_5), each of which is considered as a parallel system *per se*, work. Such a probability is

$$\begin{aligned} & \Pr(\max(X_1, X_2) = \max(X_4, X_5) = 1) \\ = & \Pr(\max(X_1, X_2) = 1) \Pr(\max(X_4, X_5) = 1) \\ = & (p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5). \end{aligned}$$

Case 2. If component 3 is not working, then the whole system functions if and only if at least one of (X_1, X_4) and (X_2, X_5), each of which is considered as a series system, works. Such a probability is

$$\Pr(\max(X_1 X_4, X_2 X_5) = 1) = p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5.$$

By the law of total probability, the unconditional probability that the bridge system functions is

$$\begin{aligned} & p_3 \Pr(\max(X_1, X_2) = \max(X_4, X_5) = 1) + (1 - p_3) \Pr(\max(X_1 X_4, X_2 X_5) = 1) \\ = & p_3(p_1 + p_2 - p_1 p_2)(p_4 + p_5 - p_4 p_5) + (1 - p_3)(p_1 p_4 + p_2 p_5 - p_1 p_2 p_4 p_5). \end{aligned}$$

When $p_i = p$ for all $i = 1, \ldots, 5$, then the reliability of the bridge system reduces to

$$\begin{aligned} p(2p - p^2)^2 + (1 - p)(2p^2 - p^4) &= p(4p^2 - 4p^3 + p^4) + (2p^2 - 2p^3 - p^4 + p^5) \\ &= 2p^2 + 2p^3 - 5p^4 + 2p^5. \end{aligned}$$

At $p = 0.7$, the reliability of the bridge system is

$$2(0.7)^2 + 2(0.7)^3 - 5(0.7)^4 + 2(0.7)^5 = \boxed{0.8016}. \quad \textbf{(Answer: D)}$$

□

Case study: Random graphs. The knowledge statements of the syllabus specifically mention random graphs, which can be analyzed by reliability-theoretic techniques. Random graphs, however, have never been tested in Exam S.

- *Definition:* What we mean by a *random graph* is a collection of nodes, any two of which can be connected by an arc whose existence is modeled as a random phenomenon. Examples of random graphs are displayed in Figure 2.2.1. For a random graph with n nodes, there can be at most $\binom{n}{2}$ arcs.

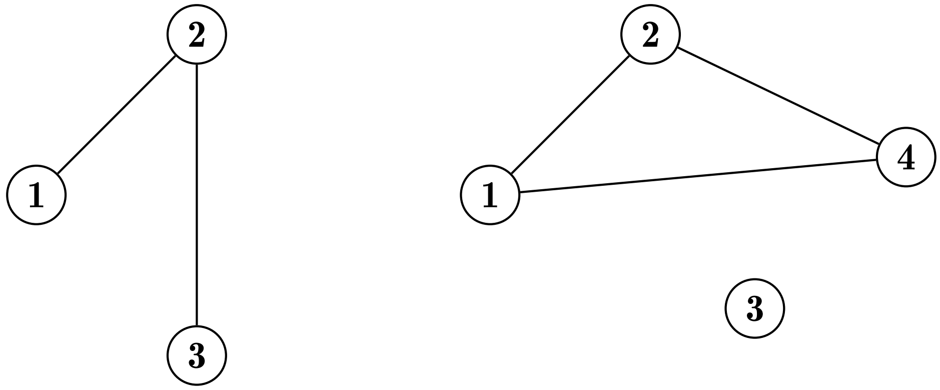


Figure 2.2.1: Two random graphs, one connected (left) and one not (right).

- *Viewing a random graph as a system:* For a given random graph, we are interested in the probability that it is *connected*. By a "connected" random graph, we mean that starting from any node of the graph, one can always travel to any other node by means of the arcs in the graph.

  To analyze this situation, we think of the random graph as a reliability system consisting of $\binom{n}{2}$ components, each of which corresponds to a potential arc in the graph. The component functions if and only if that arc is present, and the entire system functions if and only if the corresponding random graph is connected. Using this point of view, we manage to transform the probability that the graph is connected into the probability that a system functions, or equivalently, the reliability of a system.

- *Minimal cut set:* To determine the minimum cut sets of a random graph, we observe that the graph is not connected if and only if we can partition the nodes into two disjoint sets such that there is no arc from any node of the first set and any node of the second set (see, for example, the second random graph in Figure 2.2.1). For a 3-node graph, there are 3 minimal cut sets; for a 4-node graph, there are 7. It can be shown in general that there are $2^{n-1} - 1$ minimal cut sets for an n-node graph.

- *Minimal path set:* By inspection, any minimal path set of an n-node graph must contain $\boxed{n-1}$ arcs linking the n nodes without forming any cycle. For example, a minimal path set of a 3-node graph has 2 arcs (see the first random graph in Figure 2.2.1) while that of a 4-node graph has 3 arcs. It can also be shown that there are n^{n-2} distinct minimal path sets of an n-node graph.

Example 2.2.8. **(Number of minimal path sets)** Calculate the number of minimal path sets for a 4-node random graph to be connected.

A. 1

B. 2

C. 4

D. 8

E. 16

Solution. With $n = 4$, the number of minimal path sets is $4^{4-2} = \boxed{16}$. **(Answer: E)** □

- *Reliability:* Assume that the arc between node i and node j is present, independently of other arcs, with probability p_{ij}. In general, it is hard to derive explicitly the probability that a graph is connected. Ross derives, in his equation (9.9) on page 577, a recursive formula for P_n (i.e., expressed in terms of $P_1, P_2, \ldots, P_{n-1}$), the probability that that an n-node graph is connected under the assumption that each p_{ij} equals a common value p. It is doubtful whether you are expected to know this formula.

  For most purposes, it would be enough to know how to calculate P_3 (P_1 and P_2 are trivially equal to 1 and p respectively, while P_4 is too hard to calculate). This is not hard, because the probability that a 3-node graph, when viewed as a system with $\binom{3}{2} = 3$ components, is connected is the same as the reliability of a 2-out-of-3 system. This is easy to see by inspection. A rigorous proof is to consider the structure function of a 3-node graph, which is

$$\begin{aligned} \phi(\mathbf{x}) &= \max(\overbrace{x_{12}x_{23}, x_{12}x_{13}, x_{13}x_{23}}^{\text{minimal path sets}}) \\ &\stackrel{(2.1.1)}{=} 1 - (1 - x_{12}x_{23})(1 - x_{12}x_{13})(1 - x_{13}x_{23}) \\ &= x_{12}x_{13} + x_{12}x_{23} + x_{13}x_{23} - 2x_{12}x_{13}x_{23}, \end{aligned}$$

  which is the same as the structure function of a 2-out-of-3 system. Hence

$$\boxed{P_3 = p_{12}p_{13} + p_{12}p_{23} + p_{13}p_{23} - 2p_{12}p_{13}p_{23} \stackrel{(\text{if } p_{ij}=p)}{=} 3p^2 - 2p^3.} \tag{2.2.6}$$

Example 2.2.9. **(Probability of a 3-node graph being connected)** Consider a 3-node random graph with the following information:

Arc	Probability
Between node 1 and node 2	0.6
Between node 1 and node 3	0.7
Between node 2 and node 3	0.8

Calculate the probability that the graph is connected.

A. Less than 0.60

B. At least 0.60 but less than 0.70

C. At least 0.70 but less than 0.80

D. At least 0.80 but less than 0.90

E. At least 0.90

Solution. Using (2.2.6),

$$P_3 = 0.6(0.7) + 0.6(0.8) + 0.7(0.8) - 2(0.6)(0.7)(0.8) = \boxed{0.788}. \quad \textbf{(Answer: C)}$$ □

Bounds on the reliability function. As you can see from Example 2.2.7, the calculation of the reliability for a general system can be intimidating and unwieldy. It would be informative if we can obtain some easy but useful bounds on the reliability of a system. One method is to exploit the so-called *inclusion-exclusion bounds* given by

$$\begin{aligned}
\Pr\left(\bigcup_{i=1}^n E_i\right) &\le \sum_{i=1}^n \Pr(E_i), \\
\Pr\left(\bigcup_{i=1}^n E_i\right) &\ge \sum_{i=1}^n \Pr(E_i) - \sum_{1\le i<j\le n} \Pr(E_i \cap E_j), \\
\Pr\left(\bigcup_{i=1}^n E_i\right) &\le \sum_{i=1}^n \Pr(E_i) - \sum_{1\le i<j\le n} \Pr(E_i \cap E_j) + \sum_{1\le i<j<k\le n} \Pr(E_i \cap E_j \cap E_k), \\
\cdots &\ge \cdots \\
\cdots &\le \cdots
\end{aligned}$$

for any events $E_1, E_2, \ldots, E_n$. These bounds are based on the inclusion-exclusion identity

$$\begin{aligned}
\Pr\left(\bigcup_{i=1}^n E_i\right) = {} & \sum_{i=1}^n \Pr(E_i) - \sum_{1\le i<j\le n} \Pr(E_i \cap E_j) + \sum_{1\le i<j<k\le n} \Pr(E_i \cap E_j \cap E_k) \\
& - \cdots + (-1)^{n+1} \Pr(E_1 \cap E_2 \cap \cdots \cap E_n),
\end{aligned}$$

which you may have seen in your prior studies in elementary probability. The inequalities take alternative signs whenever you add one more sum into the formula.

To apply the above inequalities to estimate the reliability function of a system, we take $A_1, A_2, \ldots, A_s$ to be the minimal path sets of the system under consideration, and define

$$E_i = \{\text{all components in } A_i \text{ function}\}.$$

Then

$$\boxed{\Pr(E_i) = \prod_{l\in A_i} p_l, \quad \Pr(E_i \cap E_j) = \prod_{l\in A_i\cup A_j} p_l, \quad \text{and so forth.}}$$

Since the system works if and only if there is at least one minimal path set in which all components are functioning, we have

$$r(\mathbf{p}) = \Pr\left(\bigcup_{i=1}^{s} E_i\right),$$

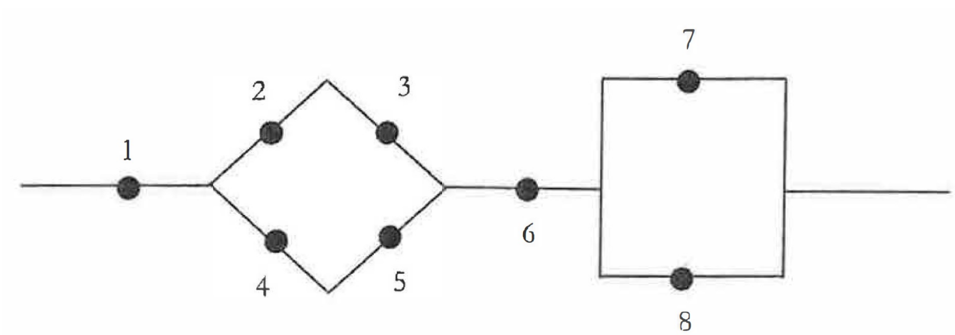
to which the inclusion-exclusion bounds can be applied. Most likely you will be asked in an exam question to use the *first two* inclusion-exclusion bounds

$$B_1 := \sum_{i=1}^{s} \Pr(E_i) \qquad \text{and} \qquad B_2 := \sum_{i=1}^{s} \Pr(E_i) - \sum_{1\le i<j\le s} \Pr(E_i \cap E_j),$$

so that

$$\boxed{\sum_{i=1}^{s} \Pr(E_i) - \sum_{1\le i<j\le s} \Pr(E_iE_j) \le r(\mathbf{p}) \le \sum_{i=1}^{s} \Pr(E_i).}$$

Example 2.2.10. (CAS Exam S Spring 2017 Question 9: Estimating the reliability of a hybrid system) You are given a system whose structure is represented in the diagram below:



The structure has identical components, with probabilities of working all equal to 0.6.

u is the upper bound reliability of the system by using the first two inclusion-exclusion bounds, defining the events in terms of minimal path sets.

Calculate u.

A. Less than 0.150

B. At least 0.150, but less than 0.250

C. At least 0.250, but less than 0.350

D. At least 0.350, but less than 0.450

E. At least 0.450

Solution. We begin by determining all the minimal path sets of the hybrid system, which, by inspection, are

$$\{1,2,3,6,7\}, \quad \{1,4,5,6,7\}, \quad \{1,2,3,6,8\}, \quad \{1,4,5,6,8\}.$$

With $s = 4$ and $\Pr(E_i) = 0.6^5 = 0.07776$ for all $i = 1, 2, 3, 4$, an upper bound on the reliability of the system by using the first two inclusion-exclusion bounds is

$$u = \sum_{i=1}^{4} \Pr(E_i) = 4(0.07776) = \boxed{0.31104}. \quad \textbf{(Answer: C)}$$ □

Remark. (i) In fact, the upper bound only relies on the first inclusion-exclusion bound.

(ii) To determine the exact reliability of the system, try Problem 2.4.17.

In the preceding examples, the inclusion-exclusion bounds are constructed based on minimal path sets. The construction can also be founded on minimal cut sets by letting $C_1, C_2, \ldots, C_r$ be the minimal cut sets of a given system and defining

$$F_i = \{\text{all components in } C_i \text{ have failed}\}.$$

Then by the definition of a minimal cut set, we have

$$\boxed{1 - r(\mathbf{p})} = \Pr\left(\bigcup_{i=1}^{r} F_i\right),$$

which is bounded above by

$$B_1' = \sum_{i=1}^{r} \Pr(F_i)$$

and below by

$$B_2' = \sum_{i=1}^{r} \Pr(F_i) - \sum_{1 \le i < j \le r} \Pr(F_i \cap F_j).$$

These imply

$$\boxed{1 - B_1' \le r(\mathbf{p}) \le 1 - B_2'.}$$

These two bounds are generally different from the inclusion-exclusion bounds based on minimal path sets.

Example 2.2.11. (CAS Exam MAS-I Fall 2019 Question 6: Bound on reliability based on minimal cut sets) You are given a system which consists of the following minimal cut sets:

$$\{1\}, \quad \{2, 3\}, \quad \{4\}, \quad \{5, 6\}$$

The system is comprised of independent and identically distributed components, each with reliability 0.9.

Calculate the lower bound of the reliability of the system by using the first two inclusion-exclusion bounds from the method of inclusion and exclusion.

A. Less than 0.75

B. At least 0.75, but less than 0.77

C. At least 0.77, but less than 0.79

D. At least 0.79, but less than 0.81

E. At least 0.81

Solution. Note that the lower bound based on minimal cut sets only requires the first inclusion-exclusion bound B_1'. Corresponding to the four minimal cut sets listed in order, we have $F_1 = F_3 = 0.1$ and $F_2 = F_4 = 0.1^2 = 0.01$. Thus $B_1' = 2(0.1) + 2(0.01) = 0.22$ and the required lower bound is $1 - B_1' = \boxed{0.78}$. **(Answer: C)** □

2.3 Expected System Lifetime

MAS-I KNOWLEDGE STATEMENT(S)

a. Joint distribution of failure times

b. Probabilities and moments

j. Effect of multiple sources of failure (multiple decrements) on failure time calculations (competing risk)

k. Non-uniform probability of component failure (multiple decrement)

OPTIONAL SYLLABUS READING(S)

Ross, Sections 9.5 and 9.6

Computing the expected system lifetime from the reliability function. In the preceding two sections, we adopt a static point of view and discuss how the reliability function of a wide variety of systems at a specific point of time can be determined. We now examine a system *dynamically* and see how the reliability function plays an important role in the determination of the expected lifetime of a system.

To begin with, note that the lifetime of the system is greater than t if and only if the system still functions at time t. The probability of the latter equals, by definition, the reliability function evaluated at the constituent survival functions (or reliability functions) at time t, i.e.,

$$\Pr(\text{system lifetime} > t) = r\left(\bar{\mathbf{F}}(t)\right) = r\left(\bar{F}_1(t), \ldots, \bar{F}_n(t)\right),$$

where r is the reliability function of the system with n components, and $\bar{F}_i(t)$ is the survival function of the i^{th} component at time t for $i = 1, \ldots, n$. Since the expected value of a non-negative random variable can be computed by integrating its survival function from 0 to ∞, we have

$$\boxed{\mathrm{E}[\text{system life}] = \int_0^\infty r\left(\bar{\mathbf{F}}(t)\right) \mathrm{d}t.} \tag{2.3.1}$$

Given the survival functions of the constituent components of a system, we can first identify its reliability function r, then use this formula to compute the mean lifetime by integration.

Examples. The calculation of the expected system lifetime can be a popular exam item. It is likely that most exam questions will involve simple systems (e.g., series, parallel, k-out-of-n with small k and n, simple hybrid, but not bridge) and easy-to-integrate component distributions (e.g., uniform, exponential, Pareto).

Example 2.3.1. (CAS Exam S Fall 2015 Question 6: Series system with i.i.d. uniform components) You are given the following information:

- An engine system consists of a series of 4 independent pumps, each of which is in operation at time zero.
- The engine system will fail when any one of the 4 pumps stops running.
- The amount of time a pump runs (in hours) is distributed uniformly over $(0, 100)$.

Calculate the expected run time of the engine system in hours.

A. Less than 17.5

B. At least 17.5, but less than 18.5

C. At least 18.5, but less than 19.5

D. At least 19.5, but less than 20.5

E. At least 20.5

Solution. From the second itemized point, the reliability function of the system is

$$r(\mathbf{p}) = p_1 p_2 p_3 p_4.$$

With $p_i = 1 - t/100$ for $i = 1, 2, 3, 4$, the expected run time of the engine system is

$$\begin{aligned}\text{E[system lifetime]} &= \int_0^{100} \left(1 - \frac{t}{100}\right)^4 \mathrm{d}t \\ &= \left[-\frac{100}{5}\left(1 - \frac{t}{100}\right)^5\right]_0^{100} \\ &= \boxed{20}. \quad \textbf{(Answer: D)}\end{aligned}$$

□

Remark. (i) This Exam S problem is a version of Example 9.26 of Ross.

(ii) In Section 9.2 of this manual, we will encounter the notion of order statistics. There we will develop shortcuts for calculating the moments of order statistics coming from a uniform distribution. With those powerful "weapons," you can complete this question very swiftly without integration by noticing that the lifetime of this series system is the first order statistics from a uniform distribution on $(0, 100)$, with an expected value of $(100)/(4 + 1) = 20$.

Example 2.3.2. **(Two-out-of-three system with Pareto components)** Consider a two-out-of-three system, in which the lifetimes of the components follow independent Pareto distributions with parameters $\alpha = 3$ and $\theta = 100$.

Calculate the expected lifetime of the system.

A. Less than 20

B. At least 20, but less than 30

C. At least 30, but less than 40

D. At least 40, but less than 50

E. At least 50

Solution. Recall that the reliability function of a two-out-of-three system is given by

$$r(\mathbf{p}) = p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3 \overset{(p_{ij}=p)}{=} 3p^2 - 2p^3.$$

With the Pareto distribution with $\alpha = 3$ and $\theta = 100$, the constituent survival function is

$$\bar{F}(t) = \left(\frac{\theta}{t+\theta}\right)^{\alpha} = \left(\frac{100}{t+100}\right)^{3}, \quad t \geq 0,$$

so the use of (2.3.1) results in

$$\begin{aligned} \text{E[system life]} &= \int_0^\infty \left[3\left(\frac{100}{t+100}\right)^6 - 2\left(\frac{100}{t+100}\right)^9\right] \mathrm{d}t \\ &= \left[-\frac{3(100)^6}{5(t+100)^5} + \frac{2(100)^9}{8(t+100)^8}\right]_0^\infty \\ &= \frac{3(100)}{5} - \frac{2(100)}{8} = \boxed{35}. \qquad \textbf{(Answer: C)} \end{aligned}$$

□

Example 2.3.3. **[HARDER!] (CAS Exam MAS-I Fall 2018 Question 8: Parallel system with uniform and exponential component distributions)** You are given the following information about a parallel system with two components:

- The first component has a lifetime that is uniform on $(0, 1)$
- The second component has a lifetime that is exponential with mean of 2

Determine which of the following is an expression for the expected lifetime of the system.

A. $\int_0^1 (1-t)\,\mathrm{d}t + \int_0^1 t\mathrm{e}^{-t/2}\,\mathrm{d}t + \int_1^\infty \mathrm{e}^{-t/2}\,\mathrm{d}t$

B. $\int_0^\infty (1-t)\mathrm{e}^{-t/2}\,\mathrm{d}t$

C. $\int_0^1 t\,\mathrm{d}t + \int_0^1 (1-t)(1-\mathrm{e}^{-t/2})\,\mathrm{d}t + \int_1^\infty (1-\mathrm{e}^{-t/2})\,\mathrm{d}t$

D. $\int_0^\infty e^{-t/2}\,dt + \int_0^1 (1-t)\,dt - \int_1^\infty (1-t)e^{-t/2}\,dt$

E. $\int_0^1 (1-t)\,dt + \int_0^\infty e^{-t/2}\,dt - \int_0^\infty (1-t)e^{-t/2}\,dt$

Solution. The reliability function of a two-component parallel system is

$$r(p_1, p_2) = p_1 + p_2 - p_1 p_2.$$

With $p_1 = \bar{F}_1(t) = 1-t$ for $t \in [0,1]$ (and 0 for $t > 1$) and $p_2 = e^{-t/2}$ for $t \geq 0$, the expected lifetime of the system can be written as

$$\begin{aligned}
\text{E[system lifetime]} &= \int_0^\infty [\overbrace{\bar{F}_1(t)}^{=0 \text{ for } t>1} + \bar{F}_2(t) - \overbrace{\bar{F}_1(t)}^{=0 \text{ for } t>1} \times \bar{F}_2(t)]\,dt \\
&= \int_0^1 (1-t)\,dt + \int_0^\infty e^{-t/2}\,dt - \int_0^1 (1-t)e^{-t/2}\,dt. \\
&= \int_0^1 (1-t)\,dt + \left(\int_0^1 e^{-t/2}\,dt + \int_1^\infty e^{-t/2}\,dt\right) - \int_0^1 (1-t)e^{-t/2}\,dt \\
&= \boxed{\int_0^1 (1-t)\,dt + \int_0^1 te^{-t/2}\,dt + \int_1^\infty e^{-t/2}\,dt}. \quad \textbf{(Answer: A)}
\end{aligned}$$

□

Remark. The CAS showed mercy by not requiring candidates to evaluate the integral $\int_0^1 te^{-t/2}\,dt$ by parts. Only a formula is required.

Special case: Exponential components. For a k-out-of-n system with *i.i.d.* exponential components with common mean θ, there is an instructive shortcut which dispenses with integration and utilizes the special properties of the exponential distribution discussed in Section 1.2 of this manual. The key is to realize that the lifetime of a k-out-of-n system can be written as

$$T_1 + \cdots + T_{n-k+1},$$

where T_i is the time between the $(i-1)^{\text{th}}$ and i^{th} failure. After $n-k+1$ failures, the number of functioning components drops to $k-1$, so by definition the k-out-of-n system fails. Now using Property 4 on page 38 concerning the minimum of independent exponential random variables, we deduce that T_1, as a minimum of n independent exponential random variables, follows an exponential distribution with a rate of n/θ (i.e., a mean of θ/n), T_2 is an exponential random variable with a rate of $(n-1)/\theta$ (i.e., a mean of $\theta/(n-1)$), and so on (the number of working components decreases by one after each failure). It follows that the expected system lifetime of the k-out-of-n system is

$$\boxed{\text{E[system life]} = \theta\left(\underbrace{\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{k}}_{n-k+1 \text{ terms}}\right)}. \tag{2.3.2}$$

Note that there are $n-k+1$ terms corresponding to the $n-k+1$ failures that trigger the failure of the whole k-out-of-n system. For instance, the expected lifetime of a 2-out-of-4 system with exponential components is

$$\theta\left(\underbrace{\frac{1}{4}+\frac{1}{3}+\frac{1}{2}}_{3\text{ terms}}\right).$$

We can go one step further and calculate the variance of the system lifetime by utilizing the independence between the T_i's (because of the memoryless property), yielding

$$\text{Var(system life)} = \theta^2\left[\frac{1}{n^2}+\frac{1}{(n-1)^2}+\cdots+\frac{1}{k^2}\right].$$

This result is presented in Exercise 9.31 of Ross.

Example 2.3.4. (CAS Exam S Spring 2017 Question 5: Series system with i.i.d. exponential components) You are given:

- R and S are two independent components in a series system.
- X and Y are the time-to-failure random variables of R and S, respectively.
- X and Y both follow the exponential distribution but with different hazard rates.
- The probability that R fails before S is 0.2.
- The mean lifetime of X is 1.
- R and S start operation at the same time.

Calculate the expected time until the system fails.

A. Less than 0.25

B. At least 0.25, but less than 0.50

C. At least 0.50, but less than 0.75

D. At least 0.75, but less than 1.00

E. At least 1.00

Solution. From the fourth and five itemized points and the exponential race probability, we are given that

$$\frac{\lambda_X}{\lambda_X+\lambda_Y}=\frac{1}{1+\lambda_Y}=0.2 \quad \Rightarrow \quad \lambda_Y=4.$$

Then the lifetime of the series system, which is the minimum of X and Y, is simply

$$1/(\lambda_X+\lambda_Y)=\boxed{0.2}. \quad \textbf{(Answer: A)}$$

□

Example 2.3.5. (CAS Exam S Fall 2017 Question 9: k-out-of-n system with i.i.d. exponential components) You are given:

- A 999-out-of-1000 system of independent identically distributed exponential components.
- The mean lifetime of each component is $\theta = 5$.

Calculate the expected system lifetime.

A. Less than 0.05

B. At least 0.05, but less than 0.10

C. At least 0.10, but less than 1.00

D. At least 1.00, but less than 10.00

E. At least 10.00

Solution. By (2.3.2) with $k = 999$ and $n = 1000$, we have

$$\text{E[system life]} = 5\left(\frac{1}{1000} + \frac{1}{999}\right) = \boxed{0.010005}. \qquad \textbf{(Answer: A)}$$ □

Example 2.3.6. [HARDER!] (CAS Exam MAS-I Spring 2018 Question 6: Parallel system with non-i.i.d. exponential components) You are given the following information about a system of only two components:

- A and B are two independent components in a parallel system.
- T_A and T_B are the time-to-failure random variables of A and B, respectively.
- T_A has the same distribution as the first waiting time of a Poisson process with rate $\lambda = 1$.
- T_B has a constant hazard rate $\lambda_B = 2$.
- A and B start operation at the same time.

Calculate the expected time until the system fails.

A. Less than 1.00

B. At least 1.00, but less than 1.25

C. At least 1.25, but less than 1.50

D. At least 1.50, but less than 1.75

E. At least 1.75

Solution. This exam question uses an indirect way to tell you that T_A and T_B are exponentially distributed with parameters $\lambda_A = 1$ and $\lambda_B = 2$, respectively. Note that because their parameters are different, T_A and T_B are *not* i.i.d., so (2.3.2) does not apply to this example.

In terms of T_A and T_B, the lifetime of the parallel system is $\max(T_A, T_B)$. There are several ways of calculating $\mathrm{E}[\max(T_A, T_B)]$.

- *Method 1 (By integration):* By (2.3.1) with $p_1 = \mathrm{e}^{-t}$ and $p_2 = \mathrm{e}^{-2t}$,

$$\begin{aligned}\mathrm{E}[\text{system lifetime}] &= \int_0^\infty (p_1 + p_2 - p_1 p_2)\,\mathrm{d}t \\ &= \int_0^\infty (\mathrm{e}^{-t} + \mathrm{e}^{-2t} - \mathrm{e}^{-3t})\,\mathrm{d}t \\ &= \frac{1}{1} + \frac{1}{2} - \frac{1}{3} \\ &= \frac{7}{6} = \boxed{1.1667}. \qquad \textbf{(Answer: B)}\end{aligned}$$

- *Method 2 (Relating "max" to "min"):* Another method is to relate $\max(T_A, T_B)$ to $\min(T_A, T_B)$ and make use of the properties of the exponential distribution we learned back in Section 1.2. Because of the identity $x + y = \min(x, y) + \max(x, y)$ for any real x and y, the maximum of T_A and T_B can be written as

$$\max(T_A, T_B) = T_A + T_B - \min(T_A, T_B).$$

Since $\min(T_A, T_B)$ is also exponentially distributed with rate $\lambda_A + \lambda_B$ (recall Property 1.2 on page 38), we have

$$\begin{aligned}\mathrm{E}[\max(T_A, T_B)] &= \mathrm{E}[T_A] + \mathrm{E}[T_B] - \mathrm{E}[\min(T_A, T_B)] \\ &= \frac{1}{\lambda_A} + \frac{1}{\lambda_B} - \frac{1}{\lambda_A + \lambda_B} \\ &= \frac{1}{1} + \frac{1}{2} - \frac{1}{1+2} \\ &= \frac{7}{6} = \boxed{1.1667}. \qquad \textbf{(Answer: B)}\end{aligned}$$

□

Non-independent components. One important assumption underlying all of the calculations in this chapter is that the components of the system have independent lifetimes. Ross, in his Subsection 9.6.1, discusses a method to estimate the expected lifetime of a parallel system with possibly non-independent components. If X_i denotes the lifetime of the i^{th} component, Ross's method relies upon the inequality

$$\text{system life} = \max_i X_i \le c + \sum_{i=1}^n (X_i - c)_+ \text{ for any constant } c.$$

Recall that $(\cdot)_+$ is the positive part function defined by $x_+ = \max(x, 0)$. The significance of this inequality is to dominate the lifetime of the parallel system which requires information about the *joint* distribution of the lifetimes of the n components by an upper bound whose expectation only involves the *marginal* distributions of the lifetimes. It follows that

$$\text{E[system life]} \le c + \sum_{i=1}^{n} \text{E}[(X_i - c)_+] = c + \sum_{i=1}^{n} \int_c^\infty \Pr(X_i > y)\, \text{d}y. \tag{2.3.3}$$

As the inequality is true for any constant c, the most useful upper bound will be obtained by minimizing the right-hand side with respect to c. By the fundamental theorem of calculus, the optimal c^* satisfies

$$\sum_{i=1}^{n} \Pr(X_i > c^*) = 1.$$

In the special case where the X_i's have identical (not necessarily independent) distributions, the value of c^* may be explicitly determined, which can be subsequently plugged into the right-hand side of (2.3.3) to produce an upper bound on the expected system lifetime.

Example 2.3.7. (Non-independent Pareto lifetimes) In a four-component parallel system, the lifetime of each component has a Pareto distribution with parameters $\alpha = 3$ and $\theta = 100$. The lifetimes of the four components are not independent.

Calculate an upper bound on the expected lifetime of the system.

A. Less than 100

B. At least 100, but less than 110

C. At least 110, but less than 120

D. At least 120, but less than 130

E. At least 130

Solution. In this case, the optimal c^* satisfies

$$4\left(\frac{100}{c^* + 100}\right)^3 = 1,$$

giving $c^* = 58.7401052$. Plugging this value into the right-hand side of (2.3.3) and integrating, we have an upper bound of

$$\begin{aligned} c^* + 4\int_{c^*}^\infty \left(\frac{100}{y + 100}\right)^3 \text{d}y &= c^* + 4(100)^3 \left[-\frac{1}{2(y+100)^2}\right]_{c^*}^\infty \\ &= c^* + 2(100)^3 \times \frac{1}{(c^* + 100)^2} \\ &= \boxed{138.1102}. \quad \textbf{(Answer: E)} \end{aligned}$$

□

2.4 End-of-chapter Problems

Typical systems

Problem 2.4.1. **(Ross, Exercise 9.4: Basic practice with structure function)** *Please refer to the book for the question statements.*

Solution. (b) Note that components 1, 6 and at least one of $\{2, 4\}$ and $\{3, 5\}$ are necessary for the system to work. The structure function is

$$\phi(\mathbf{x}) = \boxed{x_1 \max(x_2x_4, x_3x_5)x_6}.$$

(c) The structure function is

$$\phi(\mathbf{x}) = \boxed{\max(x_1, x_2x_3)x_4}.$$

□

Remark. (i) The diagram in part (a) of the exercise is suspected to be incorrect.

(ii) These systems are so simple that the use of minimal path and cut sets to determine the structure function is not necessary.

Problem 2.4.2. **(Based on Question 8 of Fall 2017 Exam S (Example 2.1.2): k-out-of-n systems)** A 3-out-of-50 system is placed in series with a 48-out-of-50-system to form a combined system.

Calculate the number of minimal path sets for the combined system.

A. Fewer than 250,000

B. At least 250,000, but fewer than 500,000

C. At least 500,000 but fewer than 10,000,000

D. At least 10,000,000, but fewer than 25,000,000

E. At least 25,000,000

Solution. Recall from Example 2.1.2 that a parallel system has $\binom{n}{1} = n$ minimal path sets. In contrast, a series system only has 1 minimal path set, corresponding to all of the n components (recall that a series system works if and only if all components work).

Back to this example, the 3-out-of-50 system and 48-out-of-50 have $\binom{50}{3} = 19,600$ and $\binom{50}{48} = 1,225$ minimal path sets, respectively. When the two systems are put in series, a minimal path set of the overall series system can be constructed from taking a minimal path set from each of the two systems, then concatenating the two selected minimal path sets. The number of minimal path sets increases considerably to $19,600 \times 1,225 = \boxed{24,010,000}$. **(Answer: D)** □

Problem 2.4.3. (Ross, Exercise 9.8: Minimal path and cut sets of a hybrid system) *Please refer to the book for the question statements.*

Solution.

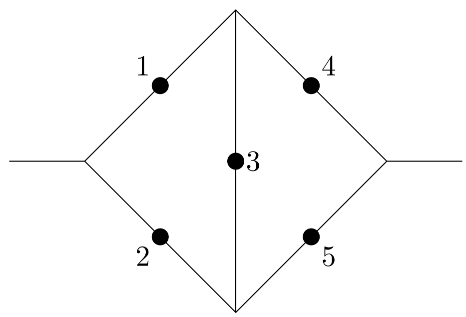
- Minimal path sets: $\boxed{\{1,3,5\},\{1,3,6\},\{2,4,5\},\{2,4,6\}}$.
- Minimal cut sets: $\boxed{\{1,2\},\{3,4\},\underbrace{\{1,4\},\{2,3\}}_{\text{Don't omit these two!}},\{5,6\}}$.

□

Remark. A by-product of this exercise is that the number of minimal path sets and the number of minimal cut sets of a system are not the same in general.

Use the following information for the next two problems.

Consider the bridge system below.



Problem 2.4.4. (Based on Example 9.8 of Ross: Identification of minimal path set for a bridge system) Determine which of the following is a minimal path set of the bridge system.

A. $\{1,2\}$

B. $\{3,5\}$

C. $\{1,2,3\}$

D. $\{2,3,4\}$

E. $\{1,2,3,4,5\}$

Solution. Only D and E guarantee that the bridge system can function when all components in D and E operate. Among D and E, only D is a minimal path set—even if components 1 and 5 are deleted from $\{1,2,3,4,5\}$, the functioning of the remaining components, namely 2, 3 and 4, is still sufficient for the functioning of the bridge system. **(Answer: D)** □

Remark. The bridge system has 4 minimal path sets: $\{1,4\},\{2,5\},\{1,3,5\}$, and $\{2,3,4\}$.

Problem 2.4.5. (Based on Example 9.9 of Ross: Identification of minimal cut set for a bridge system) Determine which of the following is a minimal cut set of the bridge system.

A. $\{1, 2\}$

B. $\{3, 5\}$

C. $\{1, 2, 3\}$

D. $\{2, 3, 4\}$

E. $\{1, 2, 3, 4, 5\}$

Solution. Only A, C, and E abort the operation of the bridge system when all components in the three sets fail. Among A, C, and E, only A is a minimal cut set—even if component 3 is dropped from $\{1, 2, 3\}$ and $\{1, 2, 3, 4, 5\}$, the failure of the remaining components is still sufficient for the failure of the bridge system. **(Answer: A)** □

Remark. A bridge system has 4 minimal cut sets: $\{1, 2\}, \{4, 5\}, \{1, 3, 5\}, \{2, 3, 4\}$.

Problem 2.4.6. [HARDER!] (Based on Exercise 9.6 of Ross: Given minimal path sets, find minimal cut sets) You are given that the minimal path sets of a system are $\{1, 2, 4\}, \{1, 3, 5\}$, and $\{5, 6\}$.

Calculate the number of minimal cut sets of the system.

A. 2

B. 3

C. 4

D. 5

E. 6

Solution. Analogous to Example 2.1.6, we choose one element from each of the four minimal path sets, paying attention to minimality, we get six minimal cut sets:

$$\begin{array}{lllll}
\{\underline{1}, 2, 4\}, & \{\underline{1}, 3, 5\}, & \{\underline{5}, 6\} & \to & \{1, 5\} \\
\{\underline{1}, 2, 4\}, & \{\underline{1}, 3, 5\}, & \{5, \underline{6}\} & \to & \{1, 6\} \\
\{1, \underline{2}, 4\}, & \{1, 3, \underline{5}\}, & \{\underline{5}, 6\} & \to & \{2, 5\} \\
\{1, 2, \underline{4}\}, & \{1, 3, \underline{5}\}, & \{\underline{5}, 6\} & \to & \{4, 5\} \\
\{1, \underline{2}, 4\}, & \{1, \underline{3}, 5\}, & \{5, \underline{6}\} & \to & \{2, 3, 6\} \\
\{1, 2, \underline{4}\}, & \{1, \underline{3}, 5\}, & \{5, \underline{6}\} & \to & \{3, 4, 6\}
\end{array}$$

(Answer: E)

□

Problem 2.4.7. **[HARDER!] (Based on Exercise 9.7 of Ross: Given minimal cut sets, find minimal path sets)** You are given that the minimal cut sets of a system are $\{1, 2, 3\}, \{2, 3, 4\}$, and $\{3, 5\}$.

Calculate the number of minimal path sets of the system.

A. 1

B. 2

C. 3

D. 4

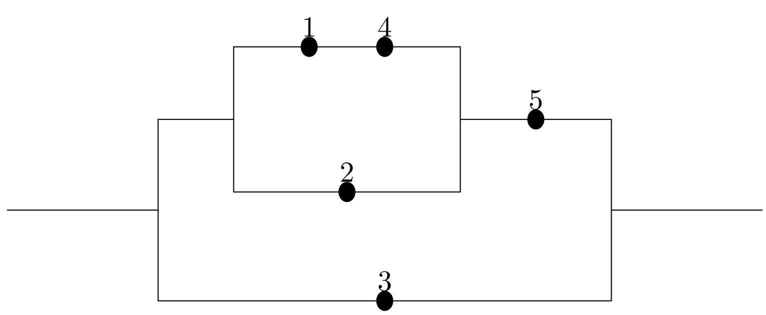
E. 5

Solution. Just as a minimal cut set must have at least one element of each minimal path set, a minimal path set must also have at least one element of each minimal cut set.

Note that $\{3\}$ is a subset of each of the three minimal cut sets, so $\{3\}$ must be a minimal path set. Excluding component 3 from the three minimal cut sets leaves $\{1, 2\}, \{2, 4\}$, and $\{5\}$. Ensuring that one element from each set is represented while ensuring minimality, we have $\{1, 4, 5\}$ and $\{2, 5\}$.

In conclusion, there are three minimal path sets, $\{3\}, \{1, 4, 5\}$, and $\{2, 5\}$. **(Answer: C)** □

Remark. Because component 3 appears in all minimal cut sets, it should play the role of a component in a parallel system. For your information, here is the structure diagram of the system:



Reliability

Problem 2.4.8. **(CAS Exam S Fall 2016 Question 10: 2-out-of-3 system)** You are given the following information about a system:

- This is a 2-out-of-3 system.
- All components are independent.
- The probability of each component functioning is $p = 0.90$.

Calculate the reliability of the system.

A. Less than 0.970

B. At least 0.970, but less than 0.975

C. At least 0.975, but less than 0.980

D. At least 0.980, but less than 0.985

E. At least 0.985

Solution. By (2.2.5), the reliability of the 2-out-of-3 system is

$$3p^2 - 2p^3 = 3(0.9)^2 - 2(0.9)^3 = \boxed{0.972}. \quad \textbf{(Answer: B)}$$

□

Problem 2.4.9. **(Ross, Exercise 9.11: Calculating reliability of a hybrid system – I)** *Please refer to the book for the question statements.*

Solution. The structure function of the system is

$$\begin{aligned}\phi(\mathbf{x}) &= \max(x_1x_3, x_2x_4)\max(x_5, x_6)\\ &= (x_1x_3 + x_2x_4 - x_1x_2x_3x_4)(x_5 + x_6 - x_5x_6).\end{aligned}$$

Replacing $\mathbf{x}$ by $\mathbf{X}$ and taking expectation lead to

$$r(\mathbf{p}) = \boxed{(p_1p_3 + p_2p_4 - p_1p_2p_3p_4)(p_5 + p_6 - p_5p_6)}.$$

□

Problem 2.4.10. **(Ross, Exercise 9.12: Calculating reliability of a hybrid system – II)** *Please refer to the book for the question statements.*

Solution. The minimal path sets are

$$\boxed{\{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}}$$

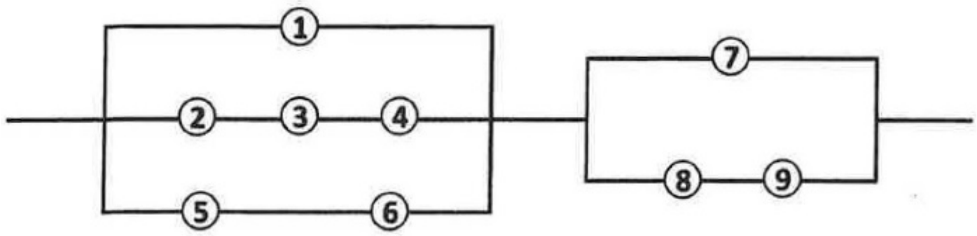
and the reliability function of the system is

$$r(\mathbf{p}) = \max(p_1, p_2, p_3)\max(p_4, p_5) = \boxed{[1 - (1-p_1)(1-p_2)(1-p_3)](p_4 + p_5 - p_4p_5)}.$$

□

Problem 2.4.11. **[HARDER!] (CAS Exam MAS-I Fall 2018 Question 6: Which component to replace to maximize reliability?)** You are given:

- A system has 9 independent components, $i = 1, 2, 3, \ldots, 9$
- $p_i = 0.75$, is the probability that the i^{th} component is functioning
- The system's structure is as pictured in the figure below:



- A new component with probability of functioning = 0.95 is available to replace one of the current components.
- The goal is to maximize the improvement of system's reliability.

Determine which one of the following components should be replaced to achieve the goal.

A. Component 1

B. Component 2

C. Component 5

D. Component 7

E. Component 8

Solution 1 (By calculations). Let's categorize the eight components into two groups:

- *Group 1:* Components 1 to 6, forming a parallel system with reliability given by

$$1-(1-p_1)(1-p_2p_3p_4)(1-p_5p_6).$$

- *Group 2:* Components 7 to 9, forming another parallel system with reliability given by

$$1-(1-p_7)(1-p_8p_9) \quad \text{or} \quad p_7+p_8p_9-p_7p_8p_9.$$

Together, the two groups form a big series system with reliability given by

$$[1-(1-p_1)(1-p_2p_3p_4)(1-p_5p_6)][p_7+p_8p_9-p_7p_8p_9].$$

Let's compute the reliability when each of the five suggested components is replaced:

Component Replaced	Reliability
1	0.8794
2	0.8453
5	0.8536
7	0.9163
8	0.8694

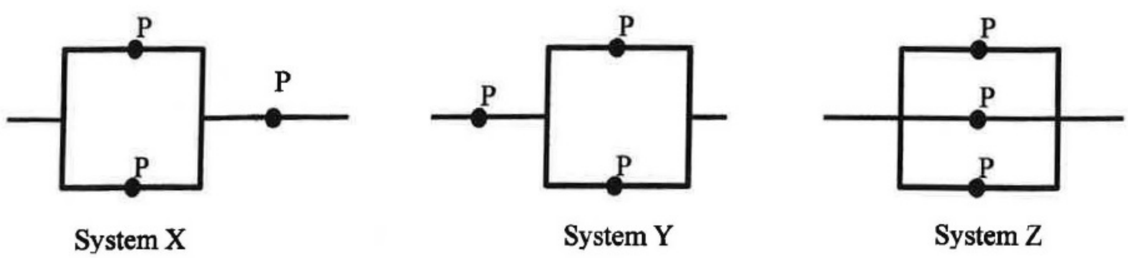
Therefore, replacing component 7 maximizes the system's reliability. **(Answer: D)** □

Solution 2 (By general reasoning). Intuitively, we should replace the component which has the "largest effect" on the operation of the system.

- *Within each parallel system:* We first notice that component 7 is more influential than components 8 (and 9). This is because component 7 alone ensures that the second parallel system can work. For component 8, the fact that it works does not guarantee the operation of the second parallel system; we also need component 9. Likewise, component 1 is more important than components 2 and 5.
- *Across the two parallel systems:* Now we compare component 1 and 7. Both components live on one branch of the two parallel systems, but the second parallel system only has two branches. Therefore, component 7 is more important than component 1. (Even if component 1 fails, we can still count on components 2, 3, 4, or components 5, 6 to function for the whole system to operate; if component 7 fails, we can only count on components 8 and 9)

Among the five suggested components, component 7 is the most influential. **(Answer: D)** □

Problem 2.4.12. (CAS Exam MAS-I Fall 2019 Question 5: Comparing the reliability of different systems) You are given the following bridge systems:



- All three systems X, Y, and Z are built with independent and identical components, P
- The reliability of System i is denoted as r_i, for $i =$ X, Y, Z

Determine which of the following best describes the reliabilities of these systems.

A. $r_X = r_Y > r_Z$

B. $r_X < r_Y < r_Z$

C. $r_X = r_Y = r_Z$

D. $r_X = r_Y < r_Z$

E. $r_X > r_Y > r_Z$

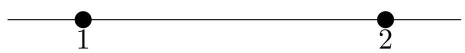
Solution. Intuitively, System Z needs only at least one of the three components to function whereas Systems X and Y require the isolated component in the series part and at least one of the two components in the parallel part to function. With this line of reasoning, we have $\boxed{r_X = r_Y < r_Z}$. **(Answer: D)** □

Remark. (i) The precise formulas for the three reliabilities are $r_X = r_Y = p(2p - p^2)$ and $r_Z = 1 - (1-p)^3$.

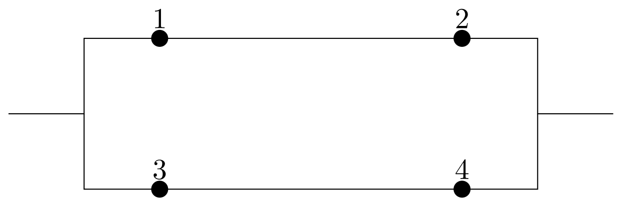
(ii) Strictly speaking, bridge systems refer to the design in Figure 2.1.4, not the hybrid design in this problem.

Problem 2.4.13. **(Ranking the reliability of several systems)** Rank the following systems in ascending order of reliability. Assume that all components operate independently with probability 0.9.

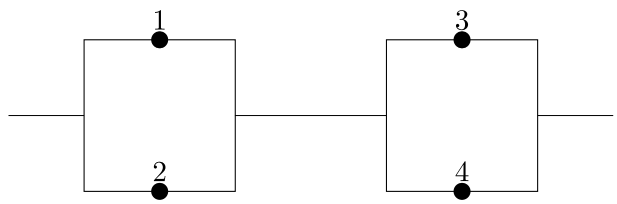
I.



II.



III.



A. I < II < III

B. I < III < II

C. II < I < III

D. II < III < I

E. III < II < I

Ambrose's comments: An ambitious exam question may require you to compare several systems with respect to reliability or rank them in order of reliability, as this problem entails. Three systems in one question!!!

Solution. Let's compute the reliability of each of the three systems.

I. This is a series system whose reliability is $0.9^2 = 0.81$.

II. The structure function is

$$\phi(\mathbf{x}) = \max(x_1x_2, x_3x_4) = 1 - (1 - x_1x_2)(1 - x_3x_4) = x_1x_2 + x_3x_4 - x_1x_2x_3x_4.$$

Taking expectations yields

$$r(p) = 2(0.9)^2 - 0.9^4 = 0.9639.$$

III. The structure function is

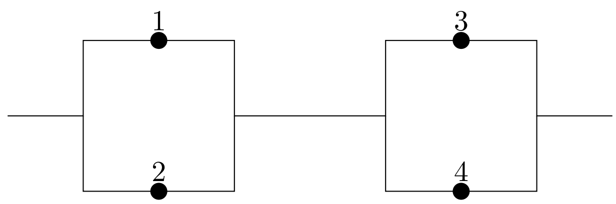
$$\phi(\mathbf{x}) = \max(x_1, x_2)\max(x_3, x_4) = (x_1 + x_2 - x_1x_2)(x_3 + x_4 - x_3x_4).$$

Taking expectations yields

$$r(p) = [2(0.9) - 0.9^2]^2 = 0.9801.$$

Therefore, the correct order is $\boxed{\text{I} < \text{II} < \text{III}}$. **(Answer: A)** □

Problem 2.4.14. **(Reliability calculations with non-i.i.d. components)** Consider the following four-component system:



Each component functions independently with the following probabilities:

Component	Probability
1	0.4
2	0.9
3	0.6
4	0.8

Calculate the reliability of the system.

A. Less than 0.6

B. At least 0.6, but less than 0.7

C. At least 0.7, but less than 0.8

D. At least 0.8, but less than 0.9

E. At least 0.9

Ambrose's comments: This is the non-i.i.d. version of Structure (II) in Example 2.4.13.

Solution. The structure function of the system is

$$\phi(\mathbf{x}) = \max(x_1, x_2)\max(x_3, x_4) = (x_1 + x_2 - x_1x_2)(x_3 + x_4 - x_3x_4).$$

Replacing the x_i's by the Bernoulli random variables X_i's and taking expectation, we have

$$\begin{aligned} \text{Reliability} &= (p_1 + p_2 - p_1p_2)(p_3 + p_4 - p_3p_4) \\ &= (0.4 + 0.9 - 0.4 \times 0.9)(0.6 + 0.8 - 0.6 \times 0.8) \\ &= \boxed{0.8648}. \quad \textbf{(Answer: D)} \end{aligned}$$

□

Problem 2.4.15. **[HARDER!] (Given the minimal cut sets, find the reliability)** You are given the following information:

- A five-component system has three minimal cut sets: $\{1, 2\}$, $\{3, 5\}$ and $\{4, 5\}$.
- The lifetimes for components 1, 2 and 3 are uniformly distributed from 0 to 5.
- The lifetimes for components 4 and 5 are exponentially distributed with mean 2.
- All components in the system are independent.
- You are starting at time 0.

Calculate the probability that the lifetime of the system will be less than 1.

A. Less than 0.20

B. At least 0.20, but less than 0.25

C. At least 0.25, but less than 0.30

D. At least 0.30, but less than 0.35

E. At least 0.35

Solution. Observe from the three minimal cut sets that the system is the same as the one in Figure 2.1.5, with structure function

$$\phi(\mathbf{x}) = (x_1 + x_2 - x_1x_2)(x_3x_4 + x_5 - x_3x_4x_5).$$

Now replacing x_1, x_2, x_3 by $p_1 = p_2 = p_3 = 1 - 1/5 = 0.8$ and x_4, x_5 by $p_4 = p_5 = \mathrm{e}^{-1/2} = 0.606531$, we can calculate the probability (i.e., reliability) that the system function for more than 1 unit as

$$(p_1 + p_2 - p_1p_2)(p_3p_4 + p_5 - p_3p_4p_5) = 0.7656.$$

Finally, the probability that the lifetime of the system will be less than 1 is $1 - 0.7656 = \boxed{0.2344}$. **(Answer: B)**

□

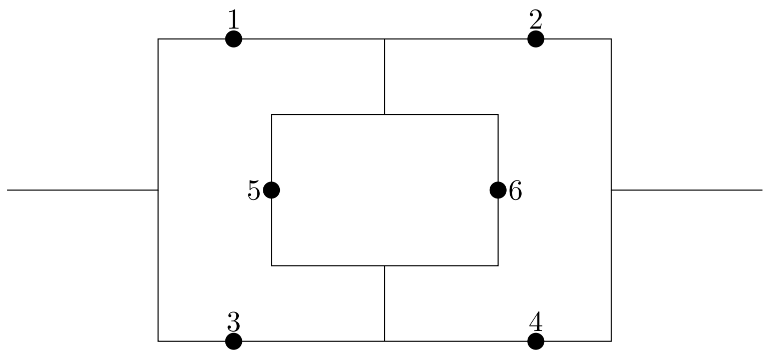
Remark. If you cannot associate by inspection the system with the one in Figure 2.1.5, you may use the series representation of parallel systems in Section 2.1 and work out the structure function of the system algebraically as

$$\begin{aligned}
\phi(\mathbf{x}) &= \max(x_1, x_2)\max(x_3, x_5)\max(x_4, x_5) \\
&= (x_1 + x_2 - x_1x_2)(x_3 + x_5 - x_3x_5)(x_4 + x_5 - x_4x_5) \\
&\overset{(x_1^2 = x_1, x_4^2 = x_4)}{=} (x_1 + x_2 - x_1x_2)(x_3x_4 + x_3x_5 - x_3x_4x_5 + \\
&\qquad x_4x_5 + x_5 - x_4x_5 - x_3x_4x_5 - x_3x_5 + x_3x_4x_5) \\
&= (x_1 + x_2 - x_1x_2)(x_3x_4 + x_5 - x_3x_4x_5).
\end{aligned}$$

This is quite laborious...

Problem 2.4.16. **[HARDER!] (A variant of bridge system)**

Consider the following six-component system:



Each component functions independently with the following probabilities:

Component	Probability
1	0.85
2	0.80
3	0.90
4	0.95
5	0.75
6	0.70

Calculate the reliability of the system.

A. Less than 0.80

B. At least 0.80, but less than 0.85

C. At least 0.85, but less than 0.90

D. At least 0.90, but less than 0.95

E. At least 0.95

Solution. Observe carefully that this system is essentially a bridge system in which the "bridge" in the middle is replaced by a two-component parallel system constructed by components 5 and 6. The same trick we used in Example 2.2.7, namely to condition on whether components 5 and 6 as a parallel system work, can be applied to evaluate the reliability of this complex system.

Case 1. If (X_5, X_6), regarded as a parallel structure, is working, then the whole system functions if and only if both of (X_1, X_3) and (X_2, X_4), each of which is considered as a parallel system, work. Such a probability is

$$\begin{aligned} & \Pr\left(\max(X_1, X_3) = \max(X_2, X_4) = 1\right) \\ = \; & \Pr\left(\max(X_1, X_3) = 1\right) \Pr\left(\max(X_2, X_4) = 1\right) \\ = \; & (0.85 + 0.90 - 0.85 \times 0.90)(0.80 + 0.95 - 0.80 \times 0.95) = 0.97515. \end{aligned}$$

Case 2. If (X_5, X_6) is not working, then the whole system functions if and only if at least one of (X_1, X_2) and (X_3, X_4), each of which is considered as a series system, works. Such a probability is

$$\begin{aligned}\Pr(\max(X_1X_2, X_3X_4) = 1) &= (p_1p_2 + p_3p_4 - p_1p_2p_3p_4)\\ &= 0.85 \times 0.80 + 0.90 \times 0.95 - 0.85(0.80)(0.90)(0.95)\\ &= 0.9536.\end{aligned}$$

The probability that (X_5, X_6) works is

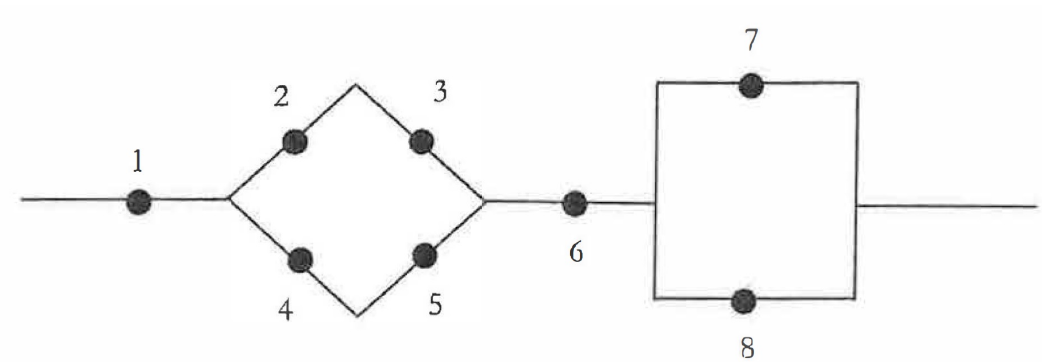
$$p_5 + p_6 - p_5p_6 = 0.75 + 0.70 - 0.75 \times 0.70 = 0.925.$$

By the law of total probability, the unconditional probability that the given system functions is

$$0.925 \times 0.97515 + (1 - 0.925) \times 0.9536 = \boxed{0.9735}. \quad \textbf{(Answer: E)}$$

□

Problem 2.4.17. **(Variant of Example 2.2.10: Estimation or exact determination of the reliability of a hybrid system)** You are given a system whose structure is represented in the diagram below:



The structure has identical components, with probabilities of working all equal to 0.8.

1. l is the lower bound reliability of the system by using the first two inclusion-exclusion bounds, defining the events in terms of minimal cut sets.

   Calculate l.

   A. Less than 0.250

   B. At least 0.250, but less than 0.350

   C. At least 0.350, but less than 0.450

   D. At least 0.450, but less than 0.550

   E. At least 0.550

Solution. By inspection, the minimal cut sets of the hybrid system are

$$C_1 = \{1\},\ C_2 = \{6\},\ C_3 = \{2,4\},\ C_4 = \{2,5\},\ C_5 = \{3,4\},\ C_6 = \{3,5\},\ C_7 = \{7,8\}.$$

With $\Pr(F_1) = \Pr(F_2) = 0.2$ and $\Pr(F_3) = \Pr(F_4) = \Pr(F_5) = \Pr(F_6) = \Pr(F_7) = 0.2^2 = 0.04$, we have

$$r(\mathbf{p}) \geq l = 1 - [2(0.2) + 5(0.04)] = \boxed{0.4}. \quad \textbf{(Answer: C)}$$ □

2. **[HARDER!]** Calculate the exact reliability of the system.

   A. Less than 0.250

   B. At least 0.250, but less than 0.350

   C. At least 0.350, but less than 0.450

   D. At least 0.450, but less than 0.550

   E. At least 0.550

   Solution. Observe that components 7 and 8 form a two-component parallel system, while components, 2, 3, 4, and 5 also constitute a two-component parallel system, where the two components are: (1) components 2 + 3, and (2): components 4 + 5. With this perspective, the hybrid system can be viewed as a four-component series system whose components are:

   (a) Component 1

   (b) Components 2, 3, 4, and 5 (a parallel system)

   (c) Component 6

   (d) Components 7 and 8 (a parallel system)

   The reliability can be written and calculated as

$$\begin{aligned} p_1(p_2p_3 + p_4p_5 - p_2p_3p_4p_5)p_6(p_7 + p_8 - p_7p_8) &\overset{(p_i=p)}{=} p^5(2-p^2)(2-p) \\ &\overset{(p=0.8)}{=} \boxed{0.5348}. \quad \textbf{(Answer: D)} \end{aligned}$$ □

Problem 2.4.18. **(Inclusion-exclusion bounds for bridge – I)** Consider a bridge system in which all components function independently with probability 0.3.

Calculate the difference between the first and second inclusion-exclusion bounds on the reliability of the bridge system based on minimal path sets.

A. Less than 0.02

B. At least 0.02, but less than 0.03

C. At least 0.03, but less than 0.04

D. At least 0.04, but less than 0.05

E. At least 0.05

Solution. The first step is to determine all possible minimal path sets of the bridge system, which we recall are

$$A_1 = \{1,4\}, \quad A_2 = \{1,3,5\}, \quad A_3 = \{2,5\}, \quad A_4 = \{2,3,4\}.$$

Then

$$\Pr(E_1) = \Pr(E_3) = p^2 \quad \text{and} \quad \Pr(E_2) = \Pr(E_4) = p^3,$$

and the first inclusion-exclusion bound is given by

$$B_1 = 2(p^2 + p^3) = 0.234$$

For the second inclusion-exclusion bound, note that among the six A_iA_j, five of them have four elements except for $A_2 \cap A_4 = \{1,2,3,4,5\}$, which has five elements, so the second inclusion-exclusion bound is

$$B_2 = 2(p^2 + p^3) - 5p^4 - p^5 = 0.191070.$$

The difference between B_1 and B_2 is $\boxed{0.0429}$. **(Answer: D)** □

Remark. (i) From Example 2.2.7, the exact value of the reliability is

$$2(0.3)^2 + 2(0.3)^3 - 5(0.3)^4 + 2(0.3)^5 = 0.1984,$$

which is quite close to the second inclusion-exclusion bound.

(ii) The inclusion-exclusion bounds based on minimal path sets work well when the component reliabilities are small.

Problem 2.4.19. **(Inclusion-exclusion bounds for bridge – II)** Consider a bridge system in which all components function independently with probability 0.7.

Calculate the difference between the first and second inclusion-exclusion bounds on the reliability of the bridge system based on minimal cut sets.

A. Less than 0.02

B. At least 0.02, but less than 0.03

C. At least 0.03, but less than 0.04

D. At least 0.04, but less than 0.05

E. At least 0.05

Solution. Let's start by determining all possible minimal cut sets of the bridge system, which are

$$C_1 = \{1,2\}, \quad C_2 = \{1,3,5\}, \quad C_3 = \{2,3,4\}, \quad C_4 = \{4,5\}.$$

Then

$$\Pr(F_1) = \Pr(F_4) = (1-p)^2 \quad \text{and} \quad \Pr(F_2) = \Pr(F_3) = (1-p)^3,$$

and the reliability function is bounded below by

$$1 - 2[(1-p)^2 + (1-p)^3] = 1 - 2[(0.3)^2 + (0.3)^3] = 0.766.$$

For the upper bound, note that among the six C_iC_j, five of them have four elements except for $C_2 \cap C_4 = \{1, 2, 3, 4, 5\}$, which has five elements, so the reliability function is bounded above by

$$0.766 + 5(0.3)^4 + (0.3)^5 = 0.808930.$$

The difference between the two bounds is $\boxed{0.0429}$. **(Answer: D)** □

Remark. (i) From Example 2.2.7, the exact value of the reliability is

$$2(0.7)^2 + 2(0.7)^3 - 5(0.7)^4 + 2(0.7)^5 = 0.80164,$$

which is quite close to the upper bound.

(ii) The inclusion-exclusion bounds based on minimal cut sets work well when the component reliabilities are large.

Expected system lifetime

Problem 2.4.20. **(CAS Exam S Spring 2017 Question 10: Series system with i.i.d. uniform components)** You are given:

- A basic series system has 3 components.
- The series system can be simplified to 2 components.
- All components are independent and function for an amount of time that is uniformly distributed over $(0, 1)$.

Calculate the increase in expected system life by simplifying from 3 components to 2 components.

A. Less than 0.03

B. At least 0.03, but less than 0.05

C. At least 0.05, but less than 0.07

D. At least 0.07, but less than 0.09

E. At least 0.09

Solution.

- When the series system has 3 components, its lifetime is the first order statistic corresponding to 3 independent $U(0, 1)$ random variables. The expected system lifetime is $\left(\frac{1}{3+1}\right)(1) = 0.25$.
- When the series system is simplified to 2 components, its lifetime is the first order statistic corresponding to only 2 independent $U(0, 1)$ random variables. The expected system lifetime is $\left(\frac{1}{2+1}\right)(1) = 1/3$.

The increase in expected system life is $1/3 - 0.25 = \boxed{0.0833}$. **(Answer: D)**

□

Problem 2.4.21. **(CAS Exam S Fall 2017 Question 10: Series vs. parallel system with i.i.d. uniform components)** You are given the following information:

- System X is a series system with two components.
- System Y is a parallel system with two components.
- All components are independent and function for an amount of time, uniformly distributed over $(0, 1)$.

Calculate the absolute value of the difference in expected system lifetimes between system X and system Y.

A. Less than 0.10

B. At least 0.10, but less than 0.20

C. At least 0.20, but less than 0.30

D. At least 0.30, but less than 0.40

E. At least 0.40

Solution.

- The lifetime of System X is the first (smaller) order statistic corresponding to 2 independent $U(0, 1)$ random variables. The expected system lifetime is $\left(\frac{1}{2+1}\right)(1) = 1/3$.
- The lifetime of System X is the second (larger) order statistic corresponding to 2 independent $U(0, 1)$ random variables. The expected system lifetime is $\left(\frac{2}{2+1}\right)(1) = 2/3$.

The absolute value of the difference in expected system lifetimes between the two systems is $1/3 = \boxed{0.3333}$. **(Answer: D)** □

Problem 2.4.22. **[HARDER!] (CAS Exam S Fall 2015 Question 7: Series system with three non-i.i.d. uniform components)** You are given the following information about a series system:

- There are three components: Component 1, Component 2, and Component 3.
- The lifetimes of Component 1 and Component 2 are each distributed U$(0, 6)$.
- The lifetime of Component 3 is distributed U$(0, 12)$.
- The component lifetimes are independent.
- U(a, b) is used to define a uniform distribution between a and b.

Calculate the expected system lifetime.

A. Less than 2.00

B. At least 2.00, but less than 2.25

C. At least 2.25, but less than 2.50

D. At least 2.50, but less than 2.75

E. At least 2.75

Solution. Because the system is a series system, its reliability function is

$$r(\mathbf{p}) = p_1 p_2 p_3.$$

The expected system lifetime is

$$\text{E[system lifetime]} = \int_0^6 \left(1 - \frac{t}{6}\right)^2 \left(1 - \frac{t}{12}\right) \mathrm{d}t.$$

To make the integration easier, we let $x = 1 - t/6$ and integrate by substitution:

$$\begin{aligned}
\text{E[system lifetime]} &= \int_0^1 x^2 \left(1 - \frac{1-x}{2}\right)(6)\,\mathrm{d}x \\
&= 3\int_0^1 x^2(1+x)\,\mathrm{d}x \\
&= 3\int_0^1 (x^2 + x^3)\,\mathrm{d}x \\
&= 3\left[\frac{x^3}{3} + \frac{x^4}{4}\right]_0^1 \\
&= \boxed{1.75}. \quad \textbf{(Answer: A)}
\end{aligned}$$

□

Remark. It is not necessary that the components of a system have identical distributions.

Problem 2.4.23. **(CAS Exam LC Fall 2014 Question 5: Parallel system with two non-i.i.d. uniform components)** For two new engines A and B, you are given the following information:

- A and B have independent future lifetimes
- The lifetime of Engine A is uniformly distributed over $(0, 20)$.
- The lifetime of Engine B is uniformly distributed over $(0, 25)$.

Calculate the expected time until both engines have failed.

A. Less than 10

B. At least 10, but less than 12

C. At least 12, but less than 14

D. At least 14, but less than 16

E. At least 16

Solution. We are interested in the expected system lifetime of the parallel system formed by engines A and B. Because the reliability of a two-component parallel system is $r(p_1, p_2) = p_1 + p_2 - p_1 p_2$, and

$$p_1 = 1 - \frac{t}{20} \text{ for } 0 \le t \le 20 \qquad \text{and} \qquad p_2 = 1 - \frac{t}{25} \text{ for } 0 \le t \le 25,$$

we have

$$\begin{aligned}
\text{E[system lifetime]} &= \int_0^{20}\left(1-\frac{t}{20}\right)\mathrm{d}t + \int_0^{25}\left(1-\frac{t}{25}\right)\mathrm{d}t - \int_0^{20}\left(1-\frac{t}{20}\right)\left(1-\frac{t}{25}\right)\mathrm{d}t \\
&= 10 + 12.5 - \frac{1}{500}\int_0^{20}(20-t)(25-t)\,\mathrm{d}t \\
&= 10 + 12.5 - \frac{1}{500}\int_0^{20}(500-45t+t^2)\,\mathrm{d}t \\
&= 10 + 12.5 - \frac{11,000/3}{500} \\
&= \boxed{15.1667}. \qquad \textbf{(Answer: D)}
\end{aligned}$$

□

Problem 2.4.24. **(Ross, Exercise 9.28: Series and parallel systems with non-i.i.d. uniform components)** *Please refer to the book for the question statements.*

Solution. For the given uniform lifetimes,

$$\bar{F}_1(t) = \begin{cases} 1, & \text{if } t < 0, \\ 1-t, & \text{if } 0 \le t \le 1, \\ 0, & \text{if } 1 < t, \end{cases} \quad \text{and} \quad \bar{F}_2(t) = \begin{cases} 1, & \text{if } t < 0, \\ 1-\frac{t}{2}, & \text{if } 0 \le t \le 2, \\ 0, & \text{if } 2 < t. \end{cases}$$

For a series system,

$$r\left(\bar{F}_1(t), \bar{F}_2(t)\right) = \bar{F}_1(t)\bar{F}_2(t) = \begin{cases} 1, & \text{if } t < 0, \\ \frac{1}{2}(1-t)(2-t), & \text{if } 0 \le t \le 1, \\ 0, & \text{if } 1 < t. \end{cases}$$

Hence

$$\text{E[system lifetime]} = \frac{1}{2}\int_0^1 (1-t)(2-t)\,\mathrm{d}t = \frac{1}{2}\left[2t - \frac{3}{2}t^2 + \frac{1}{3}t^3\right]_0^1 = \boxed{\frac{5}{12}}.$$

For a parallel system,

$$r\left(\bar{F}_1(t), \bar{F}_2(t)\right) = 1 - [1-\bar{F}_1(t)][1-\bar{F}_2(t)] = \begin{cases} 1, & \text{if } t < 0, \\ 1 - t \times t/2 = 1 - t^2/2, & \text{if } 0 \le t \le 1, \\ 1 - 1 \times t/2 = 1 - t/2, & \text{if } 1 < t \le 2, \\ 0, & \text{if } 2 < t, \end{cases}$$

and

$$\begin{aligned}
\text{E[system lifetime]} &= \int_0^1 \left(1 - \frac{t^2}{2}\right)\mathrm{d}t + \int_1^2 \left(1-\frac{t}{2}\right)\mathrm{d}t \\
&= \left[t - \frac{t^3}{6}\right]_0^1 + \left[t - \frac{t^2}{4}\right]_1^2 \\
&= \boxed{\frac{13}{12}}.
\end{aligned}$$

□

Problem 2.4.25. **(CAS Exam S Spring 2017 Question 8: Two k-out-of-n systems with i.i.d. exponential components)** You are given the following information:

- Engineers are comparing two systems.
- System U is a 4-out-of-5 system consisting of 5 identical components.
- For each component of System U, its lifetime follows the exponential distribution with a hazard rate of 0.5.
- System V is a series system consisting of 10 identical components.
- For each component of System V, its lifetime follows the exponential distribution with a hazard rate of 1.0.
- T_U and T_V are the lifetimes of Systems U and V, respectively.

Calculate the results for the expression: Min $[E[T_U], E[T_V]]$.

A. Less than 0.05

B. At least 0.05, but less than 0.15

C. At least 0.15, but less than 0.25

D. At least 0.25, but less than 0.35

E. At least 0.35

Solution.

- By (2.3.2) with $k = 4$ and $n = 5$, we have

$$\mathrm{E}[T_U] = \frac{1}{0.5}\left(\frac{1}{5} + \frac{1}{4}\right) = 0.9.$$

- System V, being a series system, is even simpler to analyze. As T_V is the minimum of 10 independent exponential random variables with rate 1, T_V itself is also exponentially distributed with a rate of $1(10) = 10$. Hence $\mathrm{E}[T_V] = 1/10 = 0.1$. Alternatively, you can also use (2.3.2) with $n = k = 10$, we have

$$\mathrm{E}[T_V] = \left(\frac{1}{1}\right)\left(\frac{1}{10}\right) = 0.1.$$

The required answer is $\min(\mathrm{E}[T_U], \mathrm{E}[T_V]) = \min(0.9, 0.1) = \boxed{0.1}$. **(Answer: B)** □

Problem 2.4.26. **[HARDER!] (Parallel system with non-i.i.d. exponential components – I)**
You are given:

- A and B are two independent components in a parallel system.
- A and B start operation at the same time.
- T_A and T_B are the time-to-failure random variables of A and B, respectively.
- The hazard rates of T_A and T_B are $\lambda_A = 2$ and $\lambda_B = 4$, respectively.

Calculate the expected time until the system fails.

A. Less than 0.25

B. At least 0.25, but less than 0.50

C. At least 0.50, but less than 0.75

D. At least 0.75, but less than 1.00

E. At least 1.00

Ambrose's comments: Note that (2.3.2) is not applicable here because the lifetimes of the two components are not identically distributed – the hazard rates are different.

Solution. The lifetime of the parallel system consists of the time of the first failure, plus the time elapsed between the second failure and the first failure.

- The time of the first failure equals the minimum of T_A and T_B, which is an exponential random variable with hazard rate $\lambda_A + \lambda_B = 2 + 4 = 6$, or mean $1/6$.
- The time elapsed between the second failure and the first failure has a mixture distribution:
  - ▷ With probability $\Pr(T_A < T_B) \overset{\text{(exp. race)}}{=} \lambda_A/(\lambda_A + \lambda_B) = 1/3$, it has an exponential distribution with rate $\lambda_B = 4$.
  - ▷ With probability $\Pr(T_A > T_B) = 1 - 1/3 = 2/3$, it has an exponential distribution with rate $\lambda_A = 2$.

  Unconditionally, the expected value of the time elapsed between the second failure and the first failure is
$$\left(\frac{1}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{5}{12}.$$

The expected lifetime of the parallel system is $1/6 + 5/12 = \boxed{0.5833}$. **(Answer: C)** □

Remark. The general formula for the expected lifetime is
$$\frac{1}{\lambda_A + \lambda_B} + \frac{\lambda_A}{(\lambda_A + \lambda_B)\lambda_B} + \frac{\lambda_B}{(\lambda_A + \lambda_B)\lambda_A};$$
see Exercise 9.29 of Ross.

Problem 2.4.27. (CAS Exam LC Fall 2014 Question 6: Parallel system with possibly non-i.i.d. exponential components – II) You are given two independent lives (x) and (y) with constant forces of mortality μ and $k\mu$ respectively, where $k \geq 1$.

You learn that the expected time to the second death is equal to three times the expected time to the first death.

Calculate k.

A. Less than 1.5

B. At least 1.5, but less than 2.0

C. At least 2.0, but less than 2.5

D. At least 2.5, but less than 3.0

E. At least 3.0

Solution.

- The expected time to the first death (series system) is $1/(\mu+k\mu) = [\mu(1+k)]^{-1}$.
- By conditioning on which life (system) dies first, we can calculate the expected time to the second death (parallel system) as

$$\begin{aligned}\frac{1}{\mu(1+k)} + \frac{\mu}{\mu(1+k)} \times \frac{1}{k\mu} + \frac{k\mu}{\mu(1+k)} \times \frac{1}{\mu} &= \frac{1}{\mu(1+k)} + \frac{1}{1+k} \times \frac{1}{k\mu} + \frac{k}{1+k} \times \frac{1}{\mu} \\ &= \frac{1+k+1/k}{\mu(1+k)}.\end{aligned}$$

We are given that

$$\frac{3}{\mu(1+k)} = \frac{1+k+1/k}{\mu(1+k)} \quad \Rightarrow \quad k^2 - 2k + 1 = (k-1)^2 = 1,$$

which means $k = \boxed{1}$, i.e., the two lives (or systems) are i.i.d. **(Answer: A)** □

Part IV

Practice Examinations

Prelude

Now that you have been well "trained" on this study manual, you need to be exposed to "unseen tests" to avoid overfitting and identify areas in which you need more "training." To this end, here are three (3) comprehensive practice exams designed to assess your understanding of the whole MAS-I exam syllabus. These practice exams have the following characteristics:

- Each exam has 45 multiple-choice questions distributed in line with the weights of the three domains in the MAS-I exam syllabus.

  As mentioned in the preface of this manual, the exam effective from Fall 2023 will consist of a variety of item types, including multiple-choice, multiple selection, point and click, and fill in the blank. If you haven't done so, please check out the following resources:

  ▷ Samples of new item types located at the bottom of this page:

  https://home.pearsonvue.com/cas.

  All of these questions are modified versions of past CAS exam questions (Exams MAS-I, MAS-II, and 5).

  ▷ Some "Testing Tips for New Item Types" available from

  https://home.pearsonvue.com/Clients/Casualty-Actuarial-Society/Testing-Tips-for-new-item-types.aspx.

  Instead of giving exam-taking tips, this file merely provides additional information about the new item types.

  As you can see, none of the new item types are fundamentally different from the traditional multiple-choice format (why did the CAS bother to introduce these new item types in the first place!? ☹), and all of them can be turned equivalently into a multiple-choice question with four or five answer choices testing the exam material in more or less the same way. To make it easy for you to score your answers using the auto-grade feature, all questions in each practice exam are multiple-choice questions with five answer choices, of which one and only one is the correct answer.

- They represent a nice combination of quantitative and qualitative exam items, as many students who took Exam MAS-I recently have suggested. The amount of calculations required by the computational questions should be reasonable (not too tedious, not trivial).

- The three exams have more or less the same level of difficulty, so you need not work them out in order. You can start with Practice Exam 3 if you like.

To make the most of these exams, here are my recommendations:

- Set aside exactly 4 hours and work on each exam in a simulated exam environment detached from distractions. Put away your manual, notes, and phone (no Facebook, Instagram, Twitter, or Snapchat for 4 hours, please!). You can only have the MAS-I tables, scratch sheets, and your calculator(s) with you. If you prefer, you may also use the CAS's scratch pad (which is effectively an Excel spreadsheet) available on the exam to do calculations.

- Budget your time wisely. Don't spend a disproportionate amount of time (say, more than 10 minutes) on a single question, no matter how difficult it seems. For a 4-hour exam with 45 questions, you should spend not more than 5.33 minutes on each question on average.

- When you are done, check your answers with the detailed illustrative solutions I provide. If you miss a question, it is important to understand the cause. Is it due to an unfamiliarity with the syllabus material, carelessness, or just bad luck? Even if you get a question right, it is beneficial to look at my solutions, which may be shorter or neater than yours.

IMPORTANT NOTE

The experiences of students who took MAS-I recently suggest that these practice exams are likely to be more difficult than the real exam, so if you do well (say, you get more than 35 out of 45 questions correct in each exam), you should be on your way to passing MAS-I with ease. Good luck!

Practice Exam 1

****BEGINNING OF EXAMINATION****

1. People arrive at a train station in accordance with a Poisson process with rate λ.

   At time 0, the train station is empty. At time 10, the bus departs.

   Five people get on the train when the bus departs.

   Calculate the expected amount of waiting time of the person who first arrives at the bus stop.

   - (A) Less than 2.5
   - (B) At least 2.5, but less than 5.0
   - (C) At least 5.0, but less than 7.5
   - (D) At least 7.5, but less than 10.0
   - (E) There is not enough information to determine the answer.

2. You are given the following information about the arrival of vehicles:

   - Taxis arrivals follow the Poisson process with rate $\lambda = 1$ per 10 minutes.
   - Bus arrivals follow the Poisson process with rate $\lambda = 4$ per 30 minutes.
   - Streetcar arrivals follow the Poisson process with rate $\lambda = 2$ per hour.

   Calculate the probability that the second vehicle arrives within 10 minutes.

   - (A) Less than 0.5
   - (B) At least 0.5, but less than 0.6
   - (C) At least 0.6, but less than 0.7
   - (D) At least 0.7, but less than 0.8
   - (E) At least 0.8

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3. Male and female customers arrive independently at a store according to Poisson processes of rate 2 per minute and 3 per minute, respectively.

Twelve females have arrived the store in ten minutes.

Calculate the expected number of customers having arrived in ten minutes.

- (A) Less than 30
- (B) At least 30, but less than 35
- (C) At least 35, but less than 40
- (D) At least 40, but less than 45
- (E) At least 45

4. At at tunnel, each arrival of a car is 4 times as likely to be a truck. The inter-arrival time of each vehicle follows an exponential distribution.

Determine the probability that the sixth vehicle that arrives at the tunnel is also the fourth car arriving.

- (A) Less than 0.2
- (B) At least 0.2, but less than 0.3
- (C) At least 0.3, but less than 0.4
- (D) At least 0.4, but less than 0.5
- (E) At least 0.5

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5. Customers enter a store in accordance with a Poisson process with rate 2 per hour. The amount of money spent by a customer is uniformly distributed over the interval $[0, 120]$.

The store operates for 10 hours per day.

Calculate the variance of the daily amount of money that the store receives.

- (A) Less than 97,000
- (B) At least 97,000, but less than 98,000
- (C) At least 98,000, but less than 99,000
- (D) At least 99,000, but less than 100,000
- (E) At least 100,000

6. The lifetime of a newly purchased electronic product is modeled by a hazard rate function given by

$$r(t) = \frac{t}{2}, \quad t > 0.$$

Calculate the expected lifetime of a newly purchased electronic product.

- (A) Less than 1.5
- (B) At least 1.5, but less than 2.0
- (C) At least 2.0, but less than 2.5
- (D) At least 2.5, but less than 3.0
- (E) At least 3.0

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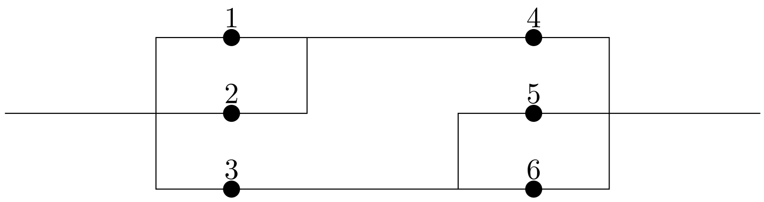
7. Consider the following six-component system:



Each component functions independently with a probability of 0.8.

Calculate the reliability of the system.

- (A) Less than 0.80
- (B) At least 0.80, but less than 0.85
- (C) At least 0.85, but less than 0.90
- (D) At least 0.90, but less than 0.95
- (E) At least 0.95

8. You are given the following information about a series system:

- The lifetime of the i^{th} component has a uniform distribution on the interval $(0, i)$ for $i = 1, 2, 3$.
- The component lifetimes are independent.

Calculate the expected system lifetime.

- (A) Less than 0.2
- (B) At least 0.2, but less than 0.4
- (C) At least 0.4, but less than 0.6
- (D) At least 0.6, but less than 0.8
- (E) At least 0.8

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9. You are given the following information:

- $X_1, X_2, \ldots$ are independent and identically distributed random variables with the following probability function:

k	0	1	2	3
$\Pr(X = k)$	0.1	0.3	0.2	0.4

- $Y_0 = 0$
- $Y_n = \max(X_1, \ldots, X_n)$ for $n \geq 1$

Determine the transition probability matrix of $\{Y_n, n = 1, 2, \ldots\}$ as a homogeneous Markov chain.

(A) $\mathbf{P} = \begin{pmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.2 & 0.4 \end{pmatrix}$

(B) $\mathbf{P} = \begin{pmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0.0 & 0.4 & 0.2 & 0.4 \\ 0.0 & 0.4 & 0.2 & 0.4 \\ 0.0 & 0.4 & 0.2 & 0.4 \end{pmatrix}$

(C) $\mathbf{P} = \begin{pmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0.0 & 0.4 & 0.2 & 0.4 \\ 0.0 & 0.0 & 0.6 & 0.4 \\ 0.0 & 0.0 & 0.6 & 0.4 \end{pmatrix}$

(D) $\mathbf{P} = \begin{pmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 0.0 & 0.4 & 0.2 & 0.4 \\ 0.0 & 0.0 & 0.6 & 0.4 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$

(E) $\mathbf{P} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.0 & 0.5 & 0.25 & 0.25 \\ 0.0 & 0.0 & 0.75 & 0.25 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$

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10. Consider a discrete-time Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $\{0, 1, 2\}$ and transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1/3 & 1/2 & 1/6 \\ 1/6 & 1/6 & 2/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}.$$

Calculate $\Pr(X_4 = 0, X_3 \neq 0, X_2 \neq 0 | X_1 = 0)$.

- (A) Less than 0.1
- (B) At least 0.1, but less than 0.2
- (C) At least 0.2, but less than 0.3
- (D) At least 0.3, but less than 0.4
- (E) At least 0.4

11. Consider a discrete-time Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $\{0, 1, 2, 3\}$ and transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2/5 & 1/5 & 1/5 & a \\ 1/5 & 2/5 & 1/5 & b \\ 1/5 & 1/5 & 2/5 & c \\ 1/5 & 1/5 & 1/5 & d \end{bmatrix} \end{array}.$$

The chain starts at state 0.

Calculate the expected number of transitions until the process enters state 0 again.

- (A) Less than 0.5
- (B) At least 0.5, but less than 1.0
- (C) At least 1.0, but less than 2.0
- (D) At least 2.0, but less than 3.0
- (E) At least 3.0

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12. Two players, A and B, initially have capital \$5 and \$10 respectively and play a game. At each round of the game, player A wins \$1 from B with probability 0.6, or loses \$1 to B with probability 0.4. Assume that different rounds of game are independent. The game ends when one of the players is ruined.

Calculate the probability that player A will ever have twice as much capital as player B.

- (A) Less than 0.5
- (B) At least 0.5, but less than 0.6
- (C) At least 0.6, but less than 0.7
- (D) At least 0.7, but less than 0.8
- (E) At least 0.8

13. For a certain life aged 50, mortality follows the Illustrative Life Table, except that from ages 50 to 51, mortality is three times the Illustrative Life Table.

Using an interest rate of 6%, calculate the level premium for a whole life insurance of 1,000 on this life with the death benefit payable at the end of the year of death.

- (A) Less than 16
- (B) At least 16, but less than 17
- (C) At least 17, but less than 18
- (D) At least 18, but less than 19
- (E) At least 19

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14. An actuary is using the inversion method to simulate a random variable with probability density function

$$f(x) = e^{-x} \exp(-e^{-x}), \qquad -\infty \leq x \leq \infty.$$

A random draw from the uniform distribution $(0, 1)$ is made, and its value is 0.6.

Calculate the value of the simulated random variable.

(A) Less than 0.2

(B) At least 0.2, but less than 0.4

(C) At least 0.4, but less than 0.6

(D) At least 0.6, but less than 0.8

(E) At least 0.8

15. You are given the following information:

- Losses follow an exponential distribution with mean θ.
- A random sample of 21 losses is distributed as follows:

Loss range	Frequency
$[0, 1000]$	6
$(1000, 2000]$	7
$(2000, \infty)$	8

Calculate the maximum likelihood estimate of θ.

(A) Less than 1950

(B) At least 1950, but less than 2100

(C) At least 2100, but less than 2250

(D) At least 2250, but less than 2400

(E) At least 2400

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16. The number of accidents on any day at a certain dangerous intersection follows a Poisson distribution.

For a random sample of size 10, the sample mean is 1.5.

Let Y be the sample mean of a new sample of size 3.

Determine the maximum likelihood estimate of the probability that Y is less than 1.

- (A) Less than 0.1
- (B) At least 0.1, but less than 0.2
- (C) At least 0.2, but less than 0.3
- (D) At least 0.3, but less than 0.4
- (E) At least 0.4

17. You are given the following probability distribution:

X	$\Pr(X = x)$
0	2θ
1	θ
2	$1 - 3\theta$

A random sample of size 40 was taken with the following results:

- The value 0 was observed 18 times.
- The value 1 was observed 7 times.
- The value 2 was observed 15 times.

Calculate the estimated standard deviation of the maximum likelihood estimator of θ.

- (A) Less than 0.01
- (B) At least 0.01, but less than 0.02
- (C) At least 0.02, but less than 0.03
- (D) At least 0.03, but less than 0.04
- (E) At least 0.04

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18. Let $\bar{X}$ be the sample mean of a random sample of size n from a gamma population with known α and unknown θ.

Determine which of the following statements is/are true.

I. $\bar{X}/\alpha$ is a minimum variance unbiased estimator of θ.

II. $\bar{X}/\alpha$ is a consistent estimator of θ.

III. $\bar{X}/\alpha$ is a sufficient statistic of θ.

- (A) None are true
- (B) I and II only
- (C) I and III only
- (D) II and III only
- (E) I, II, and III

19. A sample of independent and identically distributed random variables, $\{X_1, X_2, \ldots, X_n\}$, is drawn from a distribution with the following probability density function:

$$f(x) = \frac{1}{x\theta\sqrt{2\pi}} \exp[-(\ln x)^2/2\theta^2], \qquad x > 0,$$

where θ is a positive unknown parameter.

Determine which of the following is a sufficient statistic for θ.

- (A) $\sum_{i=1}^{n} X_i$
- (B) $\sum_{i=1}^{n} X_i^2$
- (C) $\prod_{i=1}^{n} X_i^2$
- (D) $\sum_{i=1}^{n} \ln X_i$
- (E) $\sum_{i=1}^{n} (\ln X_i)^2$

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20. You are given the following information about a portfolio of policies:

- Losses follow a lognormal distribution with parameters μ and σ^2.
- A sample of 8 losses is:

$$2, \quad 3, \quad 4, \quad 4, \quad 8, \quad 15, \quad 15, \quad 25$$

- The parameters are to be estimated by percentile matching using the 30^{th} and 70^{th} smoothed sample percentiles.

Calculate the sum of the estimate of μ and the estimate of σ.

(A) Less than 1
(B) At least 1, but less than 2
(C) At least 2, but less than 3
(D) At least 3, but less than 4
(E) At least 4

21. You are given the following:

- X_1 and X_2 are independent exponential random variables with mean θ.
- H_0: $\theta = 1$
- H_1: $\theta = 2$
- The critical region is $\{X_1 + X_2 > 5\}$.

Calculate the Type I error probability.

(A) Less than 0.02
(B) At least 0.02, but less than 0.03
(C) At least 0.03, but less than 0.04
(D) At least 0.04, but less than 0.05
(E) At least 0.05

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22. Two scales, A and B, are used in a laboratory to weigh the rock specimens (in grams). A random sample of 10 rock specimens was selected and the weight of each rock specimen was obtained from each of the two scales. The data were recorded as follows.

A	12.3	17.6	19.3	11.4	28.6	10.3	23.4	16.3	12.5	24.8
B	12.6	17.7	19.2	11.3	28.8	10.8	23.6	16.2	12.6	25.0

Assume that the weights of the rock specimens on each scale follow normal distributions.

The null hypothesis is that there is no systematic difference in the weights obtained from the two scales.

Calculate the p-value of the test.

- (A) Less than 0.01
- (B) At least 0.01, but less than 0.02
- (C) At least 0.02, but less than 0.05
- (D) At least 0.05, but less than 0.10
- (E) At least 0.10

23. Two independent random samples of 200 mental patients who did not receive psychotherapy and 200 mental patients who received psychotherapy were drawn. Their mental conditions were shown in the following table.

Condition	Deteriorated	Unchanged	Improved
Therapy	50	50	100
No Therapy	80	60	60

Calculate the lower bound of the symmetric 95% confidence interval for $p_1 - p_2$, the difference between the proportion of patients with their conditions not deteriorated in the group with psychotherapy and that in the group without psychotherapy.

- (A) Less than -0.2
- (B) At least -0.2, but less than -0.1
- (C) At least -0.1, but less than 0.0
- (D) At least 0.0, but less than 0.1
- (E) At least 0.1

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24. You are given the following:

- $X_1, X_2, \ldots, X_n$ constitute a random sample of size n from a uniform distribution on the interval $[0, \theta]$.
- $\mathrm{H}_0 : \theta = 10$
- $\mathrm{H}_1 : \theta > 10$
- Significance level $\alpha = 0.05$

Calculate the smallest value of n such that the power of the uniformly most powerful test when $\theta = 15$ is at least 0.95.

(A) 7
(B) 8
(C) 9
(D) 10
(E) 11

25. An insurance company is examining its offer of reduced rates for car insurance premiums to owners of small vehicles.

Some analysts suggest that when small vehicles are involved in accidents, the chances of serious injury are higher than that for larger sized vehicles.

A random sample of 1000 accidents is classified according to the severity of the injuries and the size of the car. The results are:

	Size of Car		
	Small	Medium	Large
Fatal/Critical	235	120	60
Non-critical	300	165	120

The following null and alternative hypotheses are created:

H_0: There is no association between size of car and severity of injury.

H_1: There is association between size of car and severity of injury.

Calculate the minimum significance level at which you would reject the null hypothesis.

(A) Less than 0.02
(B) At least 0.02, but less than 0.03
(C) At least 0.03, but less than 0.04
(D) At least 0.04, but less than 0.05
(E) At least 0.05

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26. You are given a random sample of five observations:

1	2	3	5	12

You test the hypothesis that the probability density function is:

$$f(x) = \frac{2}{x^2}e^{-2/x}, \quad x > 0.$$

Calculate the Kolmogorov-Smirnov test statistic.

- (A) Less than 0.1
- (B) At least 0.1, but less than 0.2
- (C) At least 0.2, but less than 0.3
- (D) At least 0.3, but less than 0.4
- (E) At least 0.4

27. Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size 3 from an exponential distribution with mean θ.

Calculate the variance of the sample range $Y_3 - Y_1$.

- (A) Less than $1.1\theta^2$
- (B) At least $1.1\theta^2$, but less than $1.2\theta^2$
- (C) At least $1.2\theta^2$, but less than $1.3\theta^2$
- (D) At least $1.3\theta^2$, but less than $1.4\theta^2$
- (E) At least $1.4\theta^2$

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28. You are given:

- A response variable Y has the exponential distribution

$$f(y) = \frac{1}{\theta}e^{-y/\theta}, \quad y > 0,$$

where θ is a positive constant.

- The distribution of Y lies in the exponential family

$$f(y) = \exp[a(y)b(\theta) + c(\theta) + d(y)]$$

for some functions $a(y), b(\theta), c(\theta)$ and $d(y)$.

Calculate $b(2)$.

(A) Less than -1.0
(B) At least -1.0, but less than -0.8
(C) At least -0.8, but less than -0.6
(D) At least -0.6, but less than -0.4
(E) At least -0.4

29. You are given the following models which contain regression or smoothing splines:

Model	Nature of Spline	Numbers of Spline Knots	Degree of Regression Spline
A	Smoothing	6	4
B	Regression	5	5
C	Regression	9	2

Rank the effective degrees of freedom of the three models in ascending order.

(A) A < B < C
(B) B < A < C
(C) A < C < B
(D) B < C < A
(E) C < A < B

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30. Consider the multiple linear regression model

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_1 X_2 + \beta_5 X_1 X_3 + \epsilon,$$

where

$$\begin{aligned} Y &= \text{starting salary after graduation} \\ X_1 &= \text{GPA} \\ X_2 &= \text{IQ} \\ X_3 &= \text{Gender (1 for Female and 0 for Male)} \end{aligned}$$

The least squares estimates are

$$\hat{\beta}_0 = 40, \quad \hat{\beta}_1 = 20, \quad \hat{\beta}_2 = 0.07, \quad \hat{\beta}_3 = 35, \quad \hat{\beta}_4 = 0.01, \quad \hat{\beta}_5 = -10.$$

Determine which of the following statements is/are true.

I. The predicted starting salary for a female with a GPA of 4.0 and an IQ of 110 is 127.1.

II. For a fixed value of GPA and IQ, females earn more on average than males.

III. For a fixed value of GPA and IQ, males earn more on average than females provided that the GPA is high enough.

- (A) None
- (B) I and II only
- (C) I and III only
- (D) II and III only
- (E) The correct answer is not given by (A), (B), (C), or (D).

31. A sample of size 20 is fitted to a multiple linear regression model relating a response variable to five explanatory variables.

The coefficient of determination R^2 is 0.88.

Calculate F statistic used to test the hypothesis $H_0: \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$.

- (A) Less than 20
- (B) At least 20, but less than 21
- (C) At least 21, but less than 22
- (D) At least 22, but less than 23
- (E) At least 23

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32. You are given the following information about a multiple linear regression model:

(i) The model equation is $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon$.

(ii)

Observed Y's	Fitted Y's
2.44	1.83
3.63	3.82
5.13	5.81
7.27	7.80
10.57	9.79

Calculate the adjusted R^2 of this model.

(A) Less than 0.92
(B) At least 0.92, but less than 0.94
(C) At least 0.94, but less than 0.96
(D) At least 0.96, but less than 0.98
(E) At least 0.98

33. For a simple linear regression model based on ten observations, you are given:

(i) $\sum x_i = 200$

(ii) $\sum x_i^2 = 8000$

(iii) $\sum e_i^2 = 100$

Calculate the standard error of $\hat{\beta}_0$.

(A) Less than 1.6
(B) At least 1.6, but less than 1.8
(C) At least 1.8, but less than 2.0
(D) At least 2.0, but less than 2.2
(E) At least 2.2

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34. Determine which of the following statements about different resampling methods is/are true.

I. The validation set approach is a special case of k-fold cross-validation with $k = 2$.

II. The validation set estimate of the test error tends to have a higher bias than that of k-fold cross-validation when $k \geq 2$.

III. The k-fold cross-validation estimate of the test error tends to have a higher variance than that of leave-one-out cross-validation.

(A) I only

(B) II only

(C) III only

(D) I, II, and III

(E) The correct answer is not given by (A), (B), (C), or (D).

35. You are given the following data set of three observations for two features:

	Features	
Observation	X_1	X_2
1	-1	2
2	2	-2
3	-1	0

The first principal component loading vector is $(0.6464, -0.7630)$.

Calculate the absolute difference between the largest loading and the smallest loading for the first principal component.

(A) Less than 1

(B) At least 1, but less than 2

(C) At least 2, but less than 3

(D) At least 3, but less than 4

(E) At least 4

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36. You are given the following information about a logistic regression model for the probability of policyholders having a claim:

- The model uses age as a continuous explanatory variable.
- Model output:

Parameter	$\hat{\beta}$	Standard Error
Intercept	−4.8073	0.9452
Age	1.0286	0.2004

Calculate the estimated ratio of the odds of having a claim for a policyholder aged 24 to the odds for a policyholder aged 20.

- (A) Less than 15
- (B) At least 15, but less than 30
- (C) At least 30, but less than 45
- (D) At least 45, but less than 60
- (E) At least 60

37. You are given the following information about a GLM:

- The model uses four categorical explanatory variables:
  - (a) x_1 is a categorical variables with three levels.
  - (b) x_2, x_3 are categorical variables with two levels.
  - (c) x_4 is a categorical variable with six levels.
- The model also uses a continuous explanatory variable x_5 modeled with a first order polynomial.
- There is only one interaction in the model, which is between x_1 and x_5.

Determine the maximum number of parameters in this model.

- (A) Less than 13
- (B) 13
- (C) 14
- (D) 15
- (E) At least 16

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38. You are given the following table for model selection:

Model	Deviance $\Delta(= -2\ell)$	Number of Parameters (p)	AIC
Intercept + Age	A	5	435
Intercept + Vehicle Body	392	11	414
Intercept + Age + Vehicle Value	392	X	446
Intercept + Age + Vehicle Body + Vehicle Value	B	Y	501

Calculate B.

- (A) Less than 428
- (B) 428
- (C) 429
- (D) 430
- (E) At least 430

39. Determine which of the following statements about GLMs with normal responses and identity link function is/are true.

I. Deviance follows an exact chi-square distribution.

II. Deviance cannot be used directly as a goodness of fit statistic.

III. The deviance residual is the same as the Pearson residual.

- (A) I only
- (B) II only
- (C) III only
- (D) All but III
- (E) All

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40. You are given the following information for a logistic regression model to estimate the probability of a claim for a portfolio of independent policies:

- The model uses two explanatory variables:

  (a) Age group, which is treated as a continuous explanatory variable taking values of 1, 2 and 3, modeled with a second order polynomial

  (b) Sex, which is a categorical explanatory variable with two levels

- Observations:

	Sex					
	Male			**Female**		
	Age Group			**Age Group**		
Response	1	2	3	1	2	3
No Claim	20	28	30	24	28	22
Claim	8	7	3	16	13	1

- Model output:

Parameter	$\hat{\beta}$
Intercept	-1.1155
Sex	
Female	0.0000
Male	-0.4192
Age group	1.2167
(Age group)^2	-0.5412

Calculate the estimated variance of the number of claims from the male policyholders belonging to age group 2 in the portfolio.

(A) Less than 6

(B) At least 6, but less than 7

(C) At least 7, but less than 8

(D) At least 8, but less than 9

(E) At least 9

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41. Using ordinary least squares, Steve has fitted a simple linear regression model to predict an individual's weight from the individual's height. It turns out that some of the individuals in the study are members of the same family and so have been exposed to the same environmental factors.

Determine which of the following statements about Steve's model is/are true.

I. The estimated standard errors of the coefficient estimates are lower than they should be.

II. Confidence and prediction intervals are narrower than they should be.

III. He may be led to erroneously conclude that height is a statistically significant predictor of weight when it is not.

- (A) I only
- (B) II only
- (C) III only
- (D) I, II, and III
- (E) The correct answer is not given by (A), (B), (C), or (D).

42. You are given $p = 8$ predictor variables and are considering three methods for selecting the best subset of variables: (An intercept is included in all linear models.)

Method	Number of Linear Models to Fit
Best subset selection	N_1
Forward stepwise selection	N_2
Backward stepwise selection	N_3

Calculate $N_1 + N_2 + N_3$.

- (A) 202
- (B) 328
- (C) 330
- (D) 584
- (E) 586

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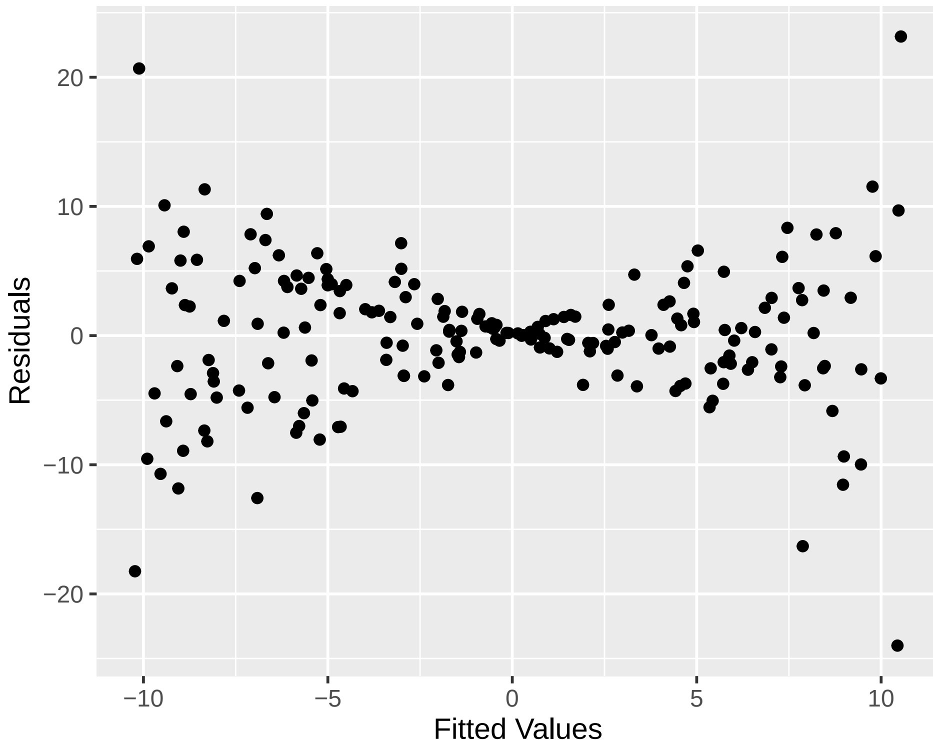
43. You have fitted a multiple linear regression model to a set of data and obtained the following residual plot:



Determine which of the following best describes the problem indicated in the residual plot.

- (A) Presence of outliers
- (B) Presence of non-linear relationships not captured by the model
- (C) Presence of high-leverage points
- (D) Heteroscedasticity
- (E) Collinearity

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44. You wish to fit a simple linear regression model to the following set of data in order to investigate the linear effect of the price of alcohol (x) on cirrhosis deaths (y). The method of least squares is used to estimate the parameters of the model.

Observation	Price of Alcohol	Cirrhosis Deaths
1	0.016	51.7
2	0.022	14.2
3	0.026	29.0
4	0.027	30.5
5	0.057	4.1
6	0.092	5.0
7	0.096	11.6

Calculate the residual of the fifth observation.

(A) -14
(B) -7
(C) 0
(D) 7
(E) 14

45. Determine which of the following statements best describes the coefficient estimates of a shrinkage method as $\lambda \to \infty$.

(A) They will approach zero.
(B) They will diverge to infinity.
(C) They will approach the ordinary least squares estimates.
(D) They will approach the local regression estimates.
(E) None of (A), (B), (C), or (D) are true.

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END OF EXAMINATION

Solutions to Practice Exam 1

Reset the Exam

Answer Key

Question	Answer
1	D
2	D
3	B
4	A
5	A
6	B
7	D
8	B
9	D
10	C
11	E
12	E
13	E
14	D
15	C

Question	Answer
16	B
17	C
18	E
19	E
20	D
21	D
22	D
23	A
24	B
25	D
26	B
27	C
28	D
29	A
30	C

Question	Answer
31	B
32	A
33	A
34	B
35	E
36	E
37	B
38	A
39	E
40	B
41	D
42	C
43	D
44	A
45	A

1. **(Conditional arrival time)**

Solution. Given that $N(10) = 5$, the conditional expected value of $S_1 (= T_1)$, the arrival time of the first person arriving at the train station, is given by

$$\mathrm{E}[S_1 | N(10)] = 5] = \frac{1}{5+1}(10) = \frac{5}{3}.$$

As the bus is scheduled to depart at time 10, the conditional expected waiting time is

$$10 - \mathrm{E}[S_1 | N(10)] = 5] = \frac{25}{3} = \boxed{8.3333}. \quad \textbf{(Answer: D)}$$

□

Remark. The value of the Poisson rate λ is not needed for solving this problem.

2. **(Modified from CAS Exam ST Fall 2014 Question 1: Probability associated with waiting time)**

Solution. The sum of the three Poisson processes remains to be a Poisson process whose rate is $1/10 + 4/30 + 2/60 = 4/15$ per minute. The probability that the second event of this combined Poisson process within the first 10 minutes is $\Pr(S_2 < 10)$, which equals $\Pr(S_2 \leq 10)$ because S_2 is a continuous random variable. The latter probability can be easily transformed into and calculated as

$$\Pr(N(10) \geq 2) = 1 - \mathrm{e}^{-4(10)/15}\left[1 + \frac{4}{15}(10)\right] = \boxed{0.7452}. \quad \textbf{(Answer: D)}$$

□

3. **(Expected value associated with thinned Poisson processes)**

Solution. The Poisson processes for male and female customers are independent, so the knowledge that 12 female customers arrive in 10 minutes does not affect the expected number of male customers in the same period, which remains at $2 \times 10 = 20$. The total expected number of customers in 10 minutes is

$$\mathrm{E}[N(10) | N^{\text{female}}(10) = 12] = \mathrm{E}[N^{\text{male}}(10)] + 12 = 20 + 12 = \boxed{32}. \quad \textbf{(Answer: B)}$$

□

4. **(Probability that the m^{th} event is of type j)**

Solution. This is the same as the probability that out of the first 5 vehicles, *exactly* 3 of them are cars, and the sixth vehicle is a car as well. This can be computed as

$$\binom{5}{3} 0.8^3 (0.2)^2 \times \underbrace{0.8}_{6^{\text{th}} \text{ vehicle is a car}} = \boxed{0.16384}. \quad \textbf{(Answer: A)}$$

□

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