ACTEX Learning

Study Manual for Exam FAM-L

2nd Edition

Johnny Li, PhD, FSA
Andrew Ng, PhD, FSA

An SOA Exam

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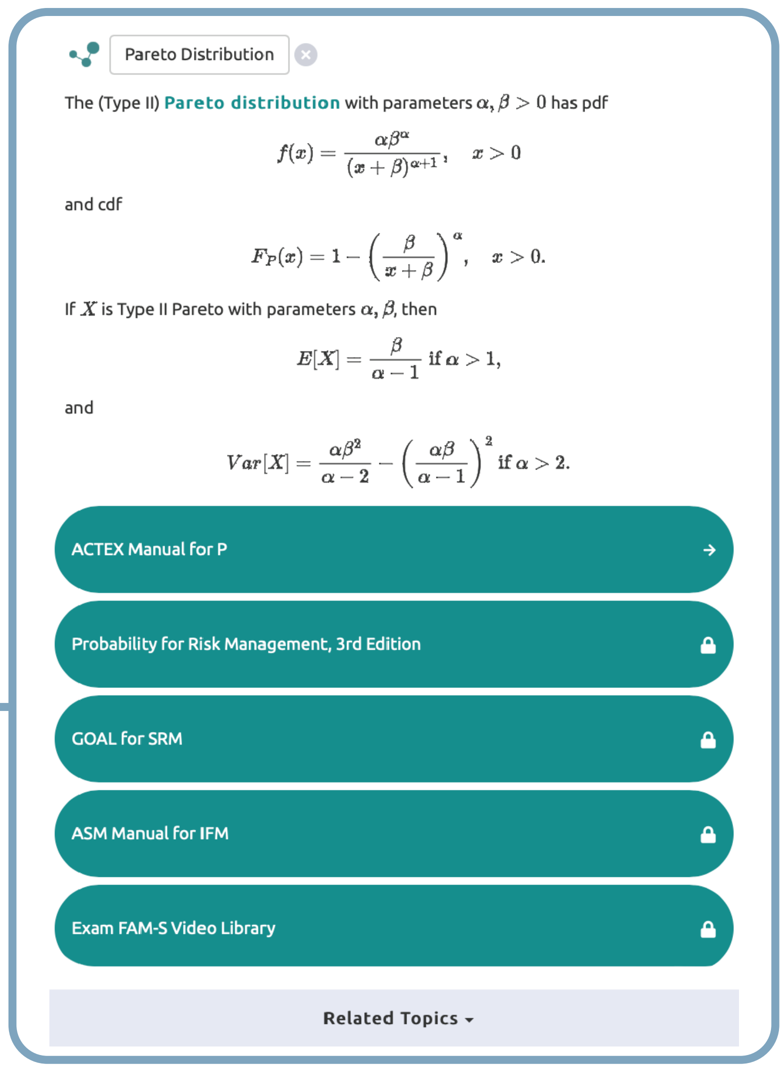

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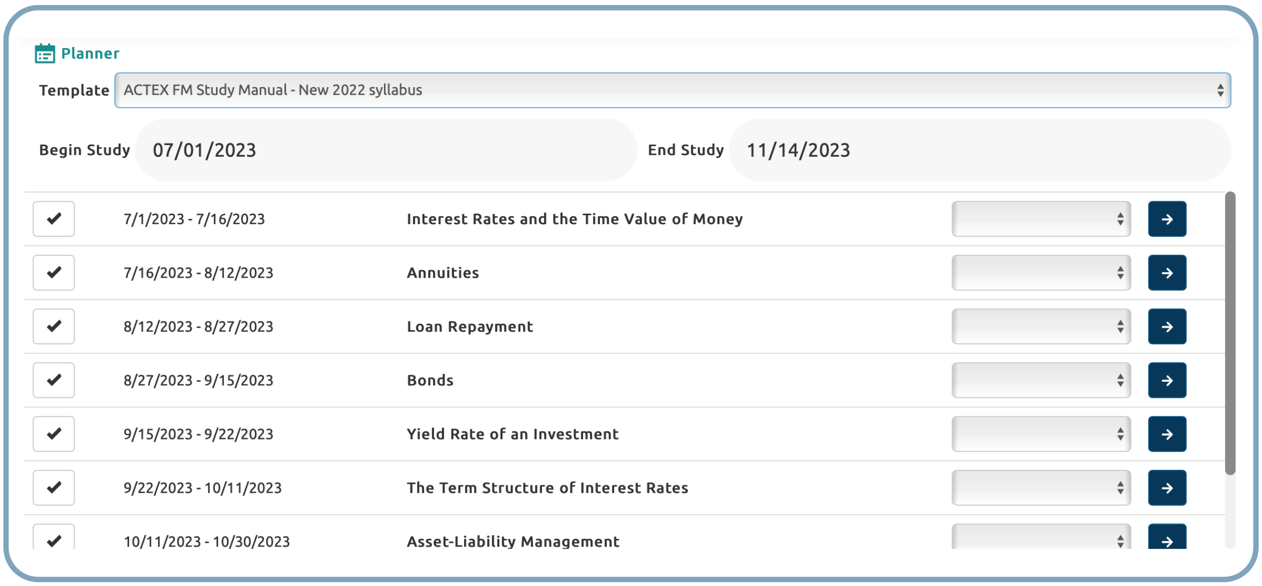
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QUESTION 19 OF 704 Question # Go! Prev Next

Question Difficulty: Advanced

An airport purchases an insurance policy to offset costs associated with excessive amounts of snowfall. The insurer pays the airport 300 for every full ten inches of snow in excess of 40 inches, up to a policy maximum of 700.

The following table shows the probability function for the random variable X of annual (winter season) snowfall, in inches, at the airport.

Inches	[0,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	[90,inf)
Probability	0.06	0.18	0.26	0.22	0.14	0.06	0.04	0.04	0.00

Calculate the standard deviation of the amount paid under the policy.

Possible Answers

A 134 | ✓ 235 | X 271 | D 313 | E 352

Help Me Start

Find the probabilities for the four possible payment amounts: 0, 300, 600, and 700.

Solution

With the amount of snowfall as X and the amount paid under the policy as Y, we have

y	$f_Y(y) = P(Y = y)$
0	$P(Y = 0) = P(0 \le X < 50) = 0.72$
300	$P(Y = 300) = P(50 \le X < 60) = 0.14$
600	$P(Y = 600) = P(60 \le X < 70) = 0.06$
700	$P(Y = 700) = P(Y \ge 70) = 0.08$

The standard deviation of Y is $\sqrt{E(Y^2) - [E(Y)]^2}$.

$$E(Y) = 0.14 \times 300 + 0.06 \times 600 + 0.08 \times 700 = 134$$

$$E(Y^2) = 0.14 \times 300^2 + 0.06 \times 600^2 + 0.08 \times 700^2 = 73400$$

$$\sqrt{E(Y^2) - [E(Y)]^2} = \sqrt{73400 - 134^2} = 235.465$$

Common Questions & Errors

Students shouldn't overthink the problem with fractional payments of 300. Also, account for probabilities in which payment cap of 700 is reached.

In these problems, we must distinguish between the REALT RV (how much snow falls) and the PAYMENT RV (when does the insurer pay)?. The problem states "The insurer pays the airport 300 for every full ten inches of snow in excess of 40 inches, up to a policy maximum of 700." So the insurer will not start paying UNTIL AFTER 10 full inches in excess of 40 inches of snow is reached (say at 50+ or 51). In other words, the insurer will pay nothing if X<50.

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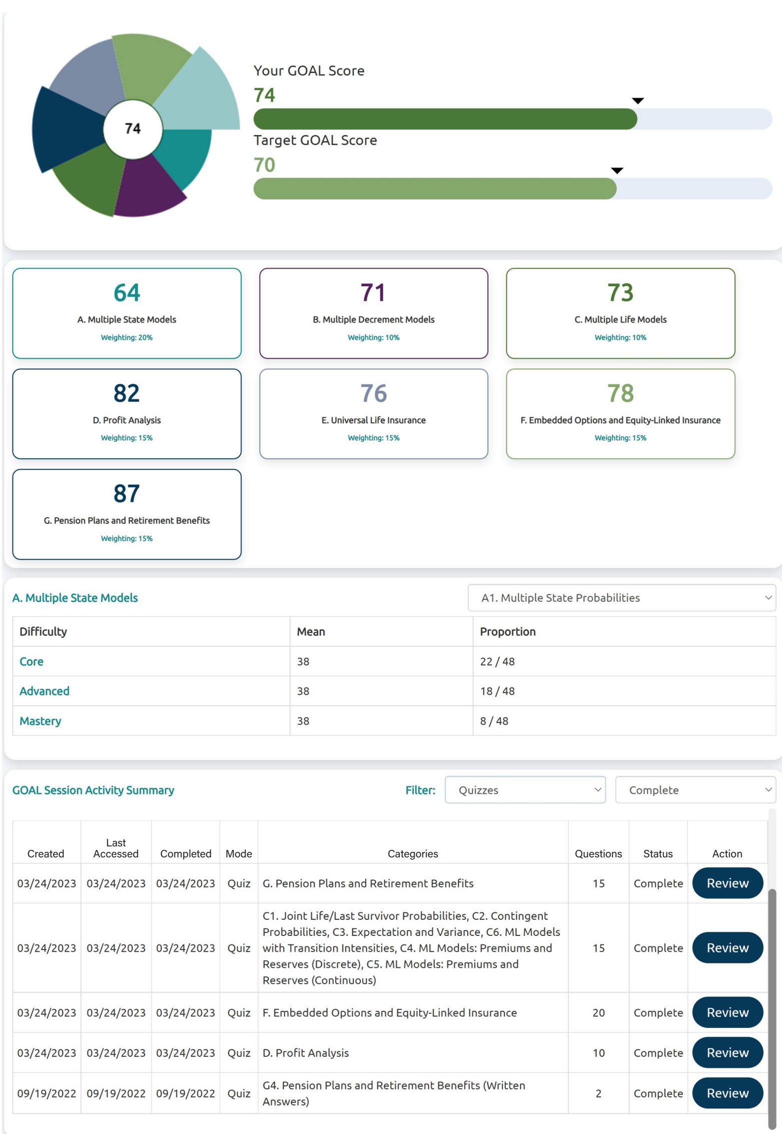

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Contents

Preface

Thank you for choosing ACTEX.

In 2022, the SoA launched Exam FAM-L (Fundamentals of Actuarial Mathematics – Long-Term). To help you prepare for this exam, ACTEX developed this brand new study manual, written by Professor Johnny Li and Andrew Ng who have deep knowledge in the exam topics.

Distinguishing features of this study manual include:

- We use graphics extensively. Graphical illustrations are probably the most effective way to explain formulas involved in Exam FAM-L. The extensive use of graphics can also help you remember various concepts and equations.

- A sleek layout is used. The font size and spacing are chosen to let you feel more comfortable in reading. Important equations are displayed in eye-catching boxes.

- Rather than splitting the manual into tiny units, each of which tells you a couple of formulas only, we have carefully grouped the exam topics into 8 chapters and 2 appendices. Such a grouping allows you to more easily identify the linkages between different concepts, which are essential for your success as multiple learning outcomes can be examined in one single exam question.

- Instead of giving you a long list of formulas, we point out which formulas are the most important. Having read this study manual, you will be able to identify the formulas you must remember and the formulas that are just variants of the key ones.

- We do not want to overwhelm you with verbose explanations. Whenever possible, concepts and techniques are demonstrated with examples and integrated into the practice problems.

You should first study all chapters of the study manual in order. Immediately after reading a chapter, do all practice problems we provide for that chapter. Make sure that you understand every single practice problem. Finally, work on the mock exams.

Before you begin your study, please download and read the exam syllabus from the SoA's website:

https://www.soa.org/education/exam-req/edu-exam-fam/

You should also check the exam home page periodically for updates, corrections or notices.

If you find a possible error in this manual, please let us know at the "Contact Us" link on the ACTEX homepage (https://actexlearning.com/contact-us). Any confirmed errata will be posted on the ACTEX website under the "Errata" link (https://actexlearning.com/errata).

Enjoy your study!

Chapter 2

Life Tables

Objectives

1. To apply life tables
2. To understand two assumptions for fractional ages: uniform distribution of death and constant force of mortality
3. To calculate moments for future lifetime random variables
4. To understand and model the effect of selection

Actuaries use spreadsheets extensively in practice. It would be very helpful if we could express survival distributions in a tabular form. Such tables, which are known as life tables, are the focus of this chapter.

2.1 Life Table Functions

Below is an excerpt of a (hypothetical) life table. In what follows, we are going to define the functions l_x and d_x, and explain how they are applied.

x	l_x	d_x
0	1000	16
1	984	7
2	977	12
3	965	75
4	890	144

In this hypothetical life table, the value of l_0 is 1,000. This starting value is called the radix of the life table. For $x = 1, 2, \ldots,$ the function l_x stands for the expected number of persons who can survive to age x. Given an assumed value of l_0, we can express any survival function $S_0(x)$ in a tabular form by using the relation

$$l_x = l_0 S_0(x).$$

In the other way around, given the life table function l_x, we can easily obtain values of $S_0(x)$ for integral values of x using the relation

$$S_0(x) = \frac{l_x}{l_0}.$$

Furthermore, we have

$$ {}_tp_x = S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{l_{x+t}/l_0}{l_x/l_0} = \frac{l_{x+t}}{l_x}, $$

which means that we can calculate ${}_tp_x$ for all integral values of t and x from the life table function l_x.

The difference $l_x - l_{x+t}$ is the expected number of deaths over the age interval of $[x, x+t)$. We denote this by ${}_td_x$. It immediately follows that ${}_td_x = l_x - l_{x+t}$.

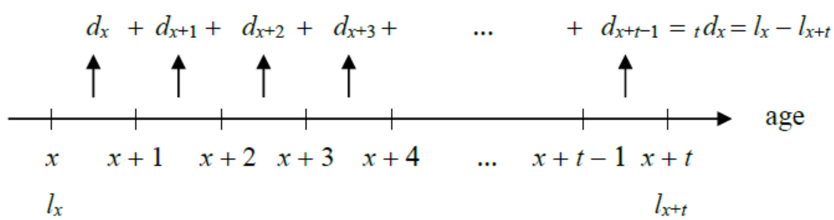
We can then calculate ${}_tq_x$ and ${}_{m|n}q_x$ by the following two relations:

$$ {}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x}, \quad {}_{m|n}q_x = \frac{{}_nd_{x+m}}{l_x} = \frac{l_{x+m} - l_{x+m+n}}{l_x}. $$

When $t = 1$, we can omit the subscript t and write ${}_1d_x$ as d_x. By definition, we have

$$ {}_td_x = d_x + d_{x+1} + \cdots + d_{x+t-1}. $$

Graphically,



Also, when $t = 1$, we have the following relations:

$$ d_x = l_x - l_{x+1}, \; p_x = \frac{l_{x+1}}{l_x}, \text{ and } q_x = \frac{d_x}{l_x}. $$

Summing up, with the life table functions l_x and d_x, we can recover survival probabilities ${}_tp_x$ and death probabilities ${}_tq_x$ for all integral values of t and x easily.

Life Table Functions

(2.1) $$ {}_tp_x = \frac{l_{x+t}}{l_x} $$

(2.2) $$ {}_td_x = l_x - l_{x+t} = d_x + d_{x+1} + \cdots + d_{x+t-1} $$

(2.3) $$ {}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x} $$

Some FAM exam and LTAM (the predecessor of FAM) questions are based on the Standard Ultimate Life Table, and some MLC (the predecessor of LTAM) exam questions are based on the Illustrative Life Table. The Standard Ultimate Life Table can be found at the SOA's website:

https://www.soa.org/4a481b/globalassets/assets/files/edu/2022/2022-10-exam-fam-l-tables.pdf,

and the Illustrative Life Table is provided in Appendix 2 of this study manual. The two tables have very similar formats. They contain a lot of information. For now, you only need to know and use the first three columns: x, l_x, and q_x (Standard Ultimate Life Table) and $1000q_x$ (Illustrative Life Table). For example, to obtain q_{43}, simply use the column labeled q_x. You should obtain $q_{43} = 0.000656$ (from the Standard Ultimate Life Table). It is also possible, but more tedious, to calculate q_{43} using the column labeled l_x; we have $q_{43} = 1 - l_{44}/l_{43} = 1 - 99104.3/99169.4 = 0.000656452$.

To get values of ${}_tp_x$ and ${}_tq_x$ for $t > 1$, you should always use the column labeled l_x. For example, we have ${}_5p_{61} = l_{66}/l_{61} = 94020.3/96305.8 = 0.976268$ and ${}_5q_{61} = 1 - {}_5p_{61} = 1 - 0.976268 = 0.023732$ (from the Standard Ultimate Life Table). Here, you should not base your calculations on the column labeled q_x, partly because that would be a lot more tedious, and partly because that may lead to a huge rounding error.

Example 2.1. You are given the following excerpt of a life table:

x	l_x	d_x
20	96178.01	99.0569
21	96078.95	102.0149
22	95976.93	105.2582
23	95871.68	108.8135
24	95762.86	112.7102
25	95650.15	116.9802

Calculate the following:

(a) ${}_5p_{20}$

(b) q_{24}

(c) ${}_{4|1}q_{20}$

Solution:

(a) ${}_5p_{20} = \dfrac{l_{25}}{l_{20}} = \dfrac{95650.15}{96178.01} = 0.994512.$

(b) $q_{24} = \dfrac{d_{24}}{l_{24}} = \dfrac{112.7102}{95762.86} = 0.001177.$

(c) ${}_{4|1}q_{20} = \dfrac{{}_1d_{24}}{l_{20}} = \dfrac{112.7102}{96178.01} = 0.001172.$

□

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Example 2.2.

You are given:

(i) $S_0(x) = 1 - \dfrac{x}{100}, \quad 0 \le x \le 100$

(ii) $l_0 = 100$

(a) Find an expression for l_x for $0 \le x \le 100$.

(b) Calculate q_2.

(c) Calculate ${}_3q_2$.

Solution:

(a) $l_x = l_0 S_0(x) = 100 - x$.

(b) $q_2 = \dfrac{l_2 - l_3}{l_2} = \dfrac{98 - 97}{98} = \dfrac{1}{98}$.

(c) ${}_3q_2 = \dfrac{l_2 - l_5}{l_2} = \dfrac{98 - 95}{98} = \dfrac{3}{98}$.

□

In Exam FAM, you may need to deal with a mixture of two populations. As illustrated in the following example, the calculation is a lot more tedious when two populations are involved.

Example 2.3.

For a certain population of 20 year olds, you are given:

(i) 2/3 of the population are nonsmokers, and 1/3 of the population are smokers.

(ii) The future lifetime of a nonsmoker is uniformly distributed over [0, 80).

(iii) The future lifetime of a smoker is uniformly distributed over [0, 50).

Calculate ${}_5p_{40}$ for a life randomly selected from those surviving to age 40.

Solution: The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 20.

- Suppose that there are l_{20} persons in the entire population initially. At time 0 (i.e., at age 20), there are $\frac{2}{3}\, l_{20}$ nonsmokers and $\frac{1}{3}\, l_{20}$ smokers.
- For nonsmokers, the proportion of individuals who can survive to age 40 is $1 - 20/80 = 3/4$. For smokers, the proportion of individuals who can survive to age 40 is $1 - 20/50 = 3/5$. At age 40, there are $\frac{3}{4}\frac{2}{3}\, l_{20} = 0.5 l_{20}$ nonsmokers and $\frac{3}{5}\frac{1}{3}\, l_{20} = 0.2 l_{20}$ smokers. Hence, among those who can survive to age 40, 5/7 are nonsmokers and 2/7 are smokers.

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Second, we need to calculate the probabilities of surviving from age 40 to age 45 for both smokers and nonsmokers.

- For a nonsmoker at age 40, the remaining lifetime is uniformly distributed over [0, 60). This means that the probability for a nonsmoker to survive from age 40 to age 45 is $1 - 5/60 = 11/12$.
- For a smoker at age 40, the remaining lifetime is uniformly distributed over [0, 30). This means that the probability for a smoker to survive from age 40 to age 45 is $1 - 5/30 = 5/6$.

Finally, for the whole population, we have

$$_5p_{40} = \frac{5}{7} \times \frac{11}{12} + \frac{2}{7} \times \frac{5}{6} = \frac{25}{28}.$$

□

2.2 Fractional Age Assumptions

We have demonstrated that given a life table, we can calculate values of $_tp_x$ and $_tq_x$ when both t and x are integers. But what if t and/or x are not integers? In this case, we need to make an assumption about how the survival function behaves between two integral ages. We call such an assumption a fractional age assumption.

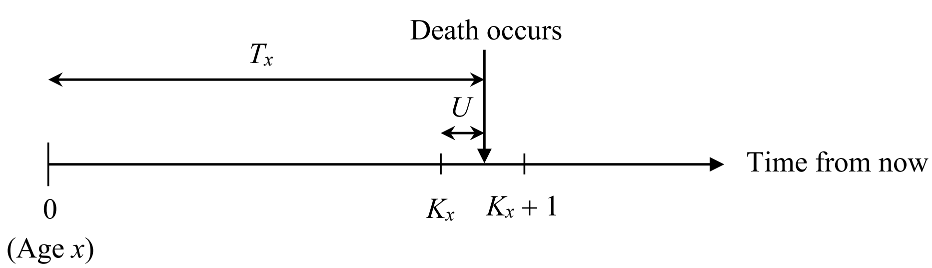
In Exam FAM, you are required to know two fractional age assumptions:

1. Uniform distribution of death
2. Constant force of mortality

We go through these assumptions one by one.

Assumption 1: Uniform Distribution of Death

The Uniform Distribution of Death (UDD) assumption is extensively used in the Exam FAM syllabus. The idea behind this assumption is that we use a bridge, denoted by U, to connect the (continuous) future lifetime random variable T_x and the (discrete) curtate future lifetime random variable K_x. The idea is illustrated diagrammatically as follows:



It is assumed that U follows a uniform distribution over the interval [0, 1], and that U and K_x are independent. Then, for $0 \leq r < 1$ and an integral value of x, we have

$$\begin{aligned} {}_rq_x &= \Pr(T_x \leq r) \\ &= \Pr(U < r \cap K_x = 0) \\ &= \Pr(U < r)\Pr(K_x = 0) \\ &= rq_x. \end{aligned}$$

The second last step follows from the assumption that U and K_x are independent, while the last step follows from the fact that U follows a uniform distribution over [0, 1].

Key Equation for the UDD Assumption

(2.4) $${}_rq_x = rq_x, \quad \text{for } 0 \leq r < 1$$

This means that under UDD, we have, for example, ${}_{0.4}q_{50} = 0.4q_{50}$. The value of q_{50} can be obtained straightforwardly from the life table. To calculate ${}_rp_x$, for $0 \leq r < 1$, we use ${}_rp_x = 1 - {}_rq_x = 1 - rq_x$. For example, we have ${}_{0.1}p_{20} = 1 - 0.1q_{20}$.

Equation (2.4) is equivalent to a linear interpolation between l_x and l_{x+1}, that is,

$$l_{x+r} = (1-r)l_x + rl_{x+1}.$$

Proof:

$$\begin{aligned} {}_rp_x &= 1 - rq_x = (1-r) + rp_x \\ \frac{l_{x+r}}{l_x} &= (1-r) + r\frac{l_{x+1}}{l_x} \\ l_{x+r} &= (1-r)l_x + rl_{x+1} \end{aligned}$$

□

You will find this equation – the interpolation between l_x and l_{x+1} – very useful if you are given a table of l_x (instead of q_x).

Application of the UDD Assumption to l_x

(2.5) $$l_{x+r} = (1-r)l_x + rl_{x+1}, \quad \text{for } 0 \leq r < 1$$

What if the subscript on the left-hand-side of ${}_rq_x$ is greater than 1? In this case, we should first use equation (1.6) from Chapter 1 to break down the probability into smaller portions. As an example, we can calculate ${}_{2.5}p_{30}$ as follows:

$${}_{2.5}p_{30} = {}_2p_{30} \times {}_{0.5}p_{32} = {}_2p_{30} \times (1 - 0.5q_{32}).$$

The value of ${}_2p_{30}$ and q_{32} can be obtained from the life table straightforwardly.

What if the subscript on the right-hand-side is not an integer? In this case, we should make use of a special trick, which we now demonstrate. Let us consider ${}_{0.1}p_{5.7}$ (both subscripts are not integers). The trick is that we multiply this probability with ${}_{0.7}p_5$, that is,

$${}_{0.7}p_5 \times {}_{0.1}p_{5.7} = {}_{0.8}p_5.$$

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This gives ${}_{0.1}p_{5.7} = \dfrac{{}_{0.8}p_5}{{}_{0.7}p_5} = \dfrac{1-0.8q_5}{1-0.7q_5}$. The value of q_5 can be obtained from the life table.

To further illustrate this trick, let us consider ${}_{3.5}p_{4.6}$: This probability can be evaluated from the following equation:

$$ {}_{0.6}p_4 \times {}_{3.5}p_{4.6} = {}_{4.1}p_4. $$

Then, we have ${}_{3.5}p_{4.6} = \dfrac{{}_{4.1}p_4}{{}_{0.6}p_4} = \dfrac{{}_4p_4\ {}_{0.1}p_8}{{}_{0.6}p_4} = \dfrac{{}_4p_4(1-0.1q_8)}{1-0.6q_4}$, and finally ${}_{3.5}q_{4.6} = 1 - \dfrac{{}_4p_4(1-0.1q_8)}{1-0.6q_4}$.

The values of ${}_4p_4$, q_8 and q_4 can be obtained from the life table.

Let us study the following example.

Example 2.4. You are given the following excerpt of a life table:

x	l_x	d_x
60	100000	300
61	99700	400
62	99300	500
63	98800	600
64	98200	700
65	97500	800

Assuming uniform distribution of deaths between integral ages, calculate the following:

(a) ${}_{0.26}p_{61}$

(b) ${}_{2.2}q_{60}$

(c) ${}_{0.3}q_{62.8}$

Solution:

(a) ${}_{0.26}p_{61} = 1 - {}_{0.26}q_{61} = 1 - 0.26 \times 400/99700 = 0.998957$.

Alternatively, we can calculate the answer by using a linear interpolation between l_{61} and l_{62} as follows:

$$ l_{61.26} = (1-0.26)l_{61} + 0.26 l_{62} = 0.74 \times 99700 + 0.26 \times 99300 = 99596. $$

It follows that ${}_{0.26}p_{61} = l_{61.26}/l_{61} = 99596/99700 = 0.998957$.

(b) ${}_{2.2}q_{60} = 1 - {}_{2.2}p_{60} = 1 - {}_2p_{60} \times {}_{0.2}p_{62} = 1 - {}_2p_{60} \times (1 - 0.2q_{62})$

$$ = 1 - \frac{l_{62}}{l_{60}}\left(1 - 0.2 \times \frac{d_{62}}{l_{62}}\right) = 1 - \frac{99300}{100000}\left(1 - 0.2 \times \frac{500}{99300}\right) = 0.008. $$

Alternatively, we can calculate the answer by using a linear interpolation between l_{62} and l_{63} as follows:

$$ l_{62.2} = (1-0.2)l_{62} + 0.2 l_{63} = 0.8 \times 99300 + 0.2 \times 98800 = 99200. $$

It follows that ${}_{2.2}q_{60} = 1 - l_{62.2}/l_{60} = 1 - 99200/100000 = 0.008$.

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(c) Here, both subscripts are non-integers, so we need to use the trick. First, we compute ${}_{0.3}p_{62.8}$ from the following equation:

$$ {}_{0.8}p_{62} \times {}_{0.3}p_{62.8} = {}_{1.1}p_{62}. $$

Trick

Rearranging the equation above, we have

$$ {}_{0.3}p_{62.8} = \frac{{}_{1.1}p_{62}}{{}_{0.8}p_{62}} = \frac{p_{62}\ {}_{0.1}p_{63}}{{}_{0.8}p_{62}} = \frac{p_{62}\,(1-0.1q_{63})}{1-0.8q_{62}} = \frac{\dfrac{98800}{99300}\left(1-0.1\times\dfrac{600}{98800}\right)}{1-0.8\times\dfrac{500}{99300}} $$
$$ = 0.998382. $$

Hence, ${}_{0.3}q_{62.8} = 1 - 0.998382 = 0.001618$.

Alternatively, we can calculate the answer by using a linear interpolation between l_{62} and l_{63} and another interpolation between l_{63} and l_{64}:

First,

$$ l_{62.8} = (1-0.8)l_{62} + 0.8l_{63} = 0.2\times 99300 + 0.8\times 98800 = 98900. $$

Second,

$$ l_{63.1} = (1-0.1)l_{63} + 0.1l_{64} = 0.9\times 98800 + 0.1\times 98200 = 98740. $$

Finally,

$$ {}_{0.3}q_{62.8} = 1 - {}_{0.3}p_{62.8} = 1 - l_{63.1}/l_{62.8} = 1 - 98740/98900 = 0.001618. $$

□

Sometimes, you may be asked to calculate the density function of T_x and the force of mortality from a life table. Under UDD, we have the following equation for calculating the density function:

$$ f_x(r) = q_x, \quad 0 \le r < 1. $$

Proof: $f_x(r) = \frac{\mathrm{d}}{\mathrm{d}r}F_x(r) = \frac{\mathrm{d}}{\mathrm{d}r}\Pr(T_x \le r) = \frac{\mathrm{d}}{\mathrm{d}r}{}_rq_x = \frac{\mathrm{d}}{\mathrm{d}r}(rq_x) = q_x.$ □

Under UDD, we have the following equation for calculating the force of mortality:

$$ \mu_{x+r} = \frac{q_x}{1-rq_x}, \quad 0 \le r < 1. $$

Proof: In general, $f_x(r) = {}_rp_x\mu_{x+r}$. Under UDD, we have $f_x(r) = q_x$ and ${}_rp_x = 1 - rq_x$. The result follows. □

Let us take a look at the following example.

Example 2.5. **[Course 3 Spring 2000 #12]**

For a certain mortality table, you are given:

(i) $\mu_{80.5} = 0.0202$

(ii) $\mu_{81.5} = 0.0408$

(iii) $\mu_{82.5} = 0.0619$

(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

(A) 0.0782 (B) 0.0785 (C) 0.0790 (D) 0.0796 (E) 0.0800

Solution: The probability that a person age 80.5 will die within two years is ${}_2q_{80.5}$. We have

$$ {}_{0.5}p_{80} \times {}_2p_{80.5} = {}_{2.5}p_{80}. $$

This gives

$$ {}_2p_{80.5} = \frac{{}_2p_{80}\ {}_{0.5}p_{82}}{{}_{0.5}p_{80}} = \frac{p_{80}p_{81}\left(1 - 0.5q_{82}\right)}{1 - 0.5q_{80}} = \frac{\left(1 - q_{80}\right)\left(1 - q_{81}\right)\left(1 - 0.5q_{82}\right)}{1 - 0.5q_{80}}. $$

We then need to find q_{80}, q_{81} and q_{82} from the information given in the question. Using $\mu_{80.5}$, we have $\mu_{80.5} = \dfrac{q_{80}}{1 - 0.5q_{80}} \Rightarrow q_{80} = 0.0200$. Similarly, by using $\mu_{81.5}$ and $\mu_{82.5}$, we obtain $q_{81} = 0.0400$ and $q_{82} = 0.0600$.

Substituting q_{80}, q_{81} and q_{82}, we obtain ${}_2p_{80.5} = 0.921794$, and hence ${}_2q_{80.5} = 1 - {}_2p_{80.5} = 0.0782$. Hence, the answer is (A).

□

Assumption 2: Constant Force of Mortality

The idea behind this assumption is that for every age x, we approximate μ_{x+r} for $0 \le r < 1$ by a constant, which we denote by $\tilde{\mu}_x$. This means

$$ \int_0^1 \mu_{x+u}\ \mathrm{d}u = \int_0^1 \tilde{\mu}_x\ \mathrm{d}u = \tilde{\mu}_x, $$

which implies $p_x = e^{-\tilde{\mu}_x}$ and $\tilde{\mu}_x = -\ln\left(p_x\right)$.

We are now ready to develop equations for calculating various death and survival probabilities. First of all, for any integer-valued x, we have

$$ {}_rp_x = \left(p_x\right)^r, \quad 0 \le r < 1. $$

Proof: ${}_rp_x = \exp\left(-\int_0^r \mu_{x+u}\mathrm{d}u\right) = \exp\left(-\int_0^r \tilde{\mu}_x\mathrm{d}u\right) = e^{-\tilde{\mu}_x r} = \left(e^{-\tilde{\mu}_x}\right)^r = \left(p_x\right)^r.$ □

For example, ${}_{0.3}p_{50} = \left(p_{50}\right)^{0.3}$, and ${}_{0.4}q_{62} = 1 - {}_{0.4}p_{62} = 1 - \left(p_{62}\right)^{0.4}$. We can generalize the equation above to obtain the following key formula.

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Key Equation for the Constant Force of Mortality Assumption

(2.6) $${}_r p_{x+u} = (p_x)^r, \quad \text{for } 0 \le r < 1 \text{ and } r + u \le 1$$

Proof: ${}_r p_{x+u} = \exp\left(-\int_0^r \mu_{x+u+t} \mathrm{d}t\right) = \exp\left(-\int_0^r \tilde{\mu}_x \mathrm{d}t\right) = e^{-\tilde{\mu}_x r} = (p_x)^r.$

[The second step follows from the fact that given $0 \le r < 1$, $u + t$ is always less than or equal to 1 when $0 \le t \le r$.] □

Notice that the key equation for the constant force of mortality assumption is based on p, while that for the UDD assumption is based on q.

This key equation means that, for example, ${}_{0.2}p_{30.3} = (p_{30})^{0.2}$. Note that the subscript u on the right-hand-side does not appear in the result, provided that the condition $r + u \le 1$ is satisfied. But what if $r + u > 1$? The answer is very simple: Split the probability! To illustrate, let us consider ${}_{0.8}p_{30.3}$. (Here, $r + u = 0.8 + 0.3 = 1.1 > 1$.) By using equation (1.6) from Chapter 1, we can split ${}_{0.8}p_{30.3}$ into two parts as follows:

$${}_{0.8}p_{30.3} = {}_{0.7}p_{30.3} \times {}_{0.1}p_{31}.$$

We intentionally consider a duration of 0.7 years for the first part, because $0.3 + 0.7 = 1$, which means we can apply the key equation ${}_r p_{x+u} = (p_x)^r$ to it. As a result, we have

$${}_{0.8}p_{30.3} = (p_{30})^{0.7} \times (p_{31})^{0.1}.$$

The values of p_{30} and p_{31} can be obtained from the life table straightforwardly.

To further illustrate, let us consider ${}_{5.6}p_{40.8}$. We can split it as follows:

$${}_{5.6}p_{40.8} = {}_{0.2}p_{40.8} \times {}_{5.4}p_{41} = {}_{0.2}p_{40.8} \times {}_{5}p_{41} \times {}_{0.4}p_{46} = (p_{40})^{0.2} \times {}_{5}p_{41} \times (p_{46})^{0.4}.$$

The values of p_{40}, ${}_5p_{41}$ and p_{46} can be obtained from the life table straightforwardly.

Interestingly, equation (2.6) implies that for $0 \le r < 1$, the value of $\ln(l_{x+r})$ can be obtained by a linear interpolation between the values of $\ln(l_x)$ and $\ln(l_{x+1})$.

Proof: Setting $u = 0$ in equation (2.6), we have

$$\begin{aligned} {}_r p_x &= (p_x)^r \\ \frac{l_{x+r}}{l_x} &= \left(\frac{l_{x+1}}{l_x}\right)^r \\ \ln(l_{x+r}) - \ln(l_x) &= r\ln(l_{x+1}) - r\ln(l_x) \\ \ln(l_{x+r}) &= (1-r)\ln(l_x) + r\ln(l_{x+1}) \end{aligned}$$

□

You will find this equation – the interpolation between $\ln(l_x)$ and $\ln(l_{x+1})$ – useful when you are given a table of l_x.

Application of the Constant Force of Mortality Assumption to l_x

$$\ln(l_{x+r}) = (1-r)\ln(l_x) + r\ln(l_{x+1}), \qquad \text{for } 0 \le r < 1$$

Example 2.6.

Assuming constant force of mortality between integral ages, repeat Example 2.4.

Solution:

(a) ${}_{0.26}p_{61} = (p_{61})^{0.26} = (99300/99700)^{0.26} = 0.998955.$

Alternatively, we can calculate the answer by interpolating between $\ln(l_{61})$ and $\ln(l_{62})$ as follows: $\ln(l_{61.26}) = (1-0.26)\ln(l_{61}) + 0.26\ln(l_{62})$, which gives $l_{61.26} = 99595.84526$. Hence, ${}_{0.26}p_{61} = l_{61.26}/l_{61} = 99595.84526/99700 = 0.998955$.

(b) ${}_{2.2}q_{60} = 1 - {}_{2.2}p_{60} = 1 - {}_{2}p_{60} \times {}_{0.2}p_{62} = 1 - {}_{2}p_{60} \times (p_{62})^{0.2}$

$$= 1 - \frac{l_{62}}{l_{60}}\left(\frac{l_{63}}{l_{62}}\right)^{0.2} = 1 - \frac{99300}{100000}\left(\frac{98800}{99300}\right)^{0.2} = 0.008002.$$

Alternatively, we can calculate the answer by interpolating between $\ln(l_{62})$ and $\ln(l_{63})$ as follows: $\ln(l_{62.2}) = (1-0.2)\ln(l_{62}) + 0.2\ln(l_{63})$, which gives $l_{62.2} = 99199.79798$. Hence, ${}_{2.2}q_{60} = 1 - l_{62.2}/l_{60} = 0.008002$.

(c) First, we consider ${}_{0.3}p_{62.8}$:

$$ {}_{0.3}p_{62.8} = {}_{0.2}p_{62.8} \times {}_{0.1}p_{63} = (p_{62})^{0.2}(p_{63})^{0.1}.$$

Hence,

$$ {}_{0.3}q_{62.8} = 1 - (p_{62})^{0.2}(p_{63})^{0.1} = 1 - \left(\frac{l_{63}}{l_{62}}\right)^{0.2}\left(\frac{l_{64}}{l_{63}}\right)^{0.1}$$

$$= 1 - \left(\frac{98800}{99300}\right)^{0.2}\left(\frac{98200}{98800}\right)^{0.1} = 0.001617.$$

Alternatively, we can calculate the answer by an interpolation between $\ln(l_{62})$ and $\ln(l_{63})$ and another interpolation between $\ln(l_{63})$ and $\ln(l_{64})$.

First, $\ln(l_{62.8}) = (1-0.8)\ln(l_{62}) + 0.8\ln(l_{63})$, which gives $l_{62.8} = 98899.79818$.

Second, $\ln(l_{63.1}) = (1-0.1)\ln(l_{63}) + 0.1\ln(l_{64})$, which gives $l_{63.1} = 98739.8354$.

Finally, ${}_{0.3}q_{62.8} = 1 - l_{63.1}/l_{62.8} = 0.001617$.

□

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Example 2.7.

You are given the following life table:

x	l_x	d_x
90	1000	50
91	950	50
92	900	60
93	840	c_1
94	c_2	70
95	700	80

(a) Find the values of c_1 and c_2.

(b) Calculate ${}_{1.4}p_{90}$, assuming uniform distribution of deaths between integer ages.

(c) Repeat (b) by assuming constant force of mortality between integer ages.

Solution:

(a) We have $840 - c_1 = c_2$ and $c_2 - 70 = 700$. This gives $c_2 = 770$ and $c_1 = 70$.

(b) Assuming uniform distribution of deaths between integer ages, we have

$$\begin{aligned}
{}_{1.4}p_{90} &= p_{90} \times {}_{0.4}p_{91} \\
&= p_{90}\,(1 - 0.4q_{91}) \\
&= \frac{l_{91}}{l_{90}}\left(1 - 0.4\frac{d_{91}}{l_{91}}\right) \\
&= \frac{950}{1000}\left(1 - 0.4 \times \frac{50}{950}\right) \\
&= 0.93.
\end{aligned}$$

Alternatively, you can compute the answer by interpolating between l_{92} and l_{91}:

$${}_{1.4}p_{90} = p_{90} \times {}_{0.4}p_{91} = \frac{l_{91}}{l_{90}}\left(\frac{0.4l_{92} + 0.6l_{91}}{l_{91}}\right) = \frac{0.4 \times 900 + 0.6 \times 950}{1000} = 0.93$$

(c) Assuming constant force of mortality between integer ages, we have

$$\begin{aligned}
{}_{1.4}p_{90} &= p_{90} \times_{0.4} p_{91} \\
&= p_{90} \times (p_{91})^{0.4} \\
&= \frac{950}{1000}\left(\frac{900}{950}\right)^{0.4} \\
&= 0.92968.
\end{aligned}$$

□

Let us conclude this section with the following table, which summarizes the formulas for the two fractional age assumptions.

	UDD	**Constant force**
${}_r p_x$	$1 - rq_x$	$(p_x)^r$
${}_r q_x$	rq_x	$1 - (p_x)^r$
μ_{x+r}	$\dfrac{q_x}{1 - rq_x}$	$-\ln(p_x)$

In the table, x is an integer and $0 \leq r < 1$. The shaded formulas are the key formulas that you must remember for the examination.

2.3 Select-and-Ultimate Tables

Insurance companies typically assess risk before they agree to insure you. They cannot stay in business if they sell life insurance to someone who has just discovered he has only a few months to live. A team of underwriters will usually review information about you before you are sold insurance (although there are special insurance types called "guaranteed issue" which cannot be underwritten). For this reason, a person who has just purchased life insurance has a lower probability of death than a person the same age in the general population. The probability of death for a person who has just been issued life insurance is called a select probability. In this section, we focus on the modeling of select probabilities.

Let us define the following notation.

- $[x]$ indicates the age at selection (i.e., the age at which the underwriting was done).
- $[x]+t$ indicates a person currently age $x+t$ who was selected at age x (i.e., the underwriting was done at age x). This implies that the insurance contract has elapsed for t years.

For example, we have the following select probabilities:

- $q_{[x]}$ is the probability that a life age x now dies before age $x+1$, given that the life is selected at age x.
- $q_{[x]+t}$ is the probability that a life age $x+t$ now dies before age $x+t+1$, given that the life was selected at age x.

Due to the effect of underwriting, a select death probability $q_{[x]+t}$ must be no greater than the corresponding ordinary death probability q_{x+t}. However, the effect of underwriting will not last forever. The period after which the effect of underwriting is completely gone is called the select period. Suppose that the select period is n years, we have

$$q_{[x]+t} < q_{x+t}, \quad \text{for } t < n.$$
$$q_{[x]+t} = q_{x+t}, \quad \text{for } t \geq n.$$

The ordinary death probability q_{x+t} is called the ultimate death probability. A life table that contains both select probabilities and ultimate probabilities is called a select-and-ultimate life table. The following is an excerpt of a (hypothetical) select-and-ultimate table with a select period of two years.

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
40	0.04	0.06	0.08	42
41	0.05	0.07	0.09	43
42	0.06	0.08	0.10	44
43	0.07	0.09	0.11	45

It is important to know how to apply such a table. Let us consider a person who is currently age 41 and is just selected. The death probabilities for this person are as follows:

Age 41: $q_{[41]} = 0.05$

Age 42: $q_{[41]+1} = 0.07$

Age 43: $q_{[41]+2} = q_{43} = 0.09$

Age 44: $q_{[41]+3} = q_{44} = 0.10$

Age 45: $q_{[41]+4} = q_{45} = 0.11$

As you see, the select-and-ultimate table is not difficult to use. We progress horizontally until we reach the ultimate death probability, then we progress vertically as when we are using an ordinary life table. To further illustrate, let us consider a person who is currently age 41 and was selected at age 40. The death probabilities for this person are as follows:

Age 41: $q_{[40]+1} = 0.06$

Age 42: $q_{[40]+2} = q_{42} = 0.08$

Age 43: $q_{[40]+3} = q_{43} = 0.09$

Age 44: $q_{[40]+4} = q_{44} = 0.10$

Age 45: $q_{[40]+5} = q_{45} = 0.11$

Even though the two persons we considered are of the same age now, their current death probabilities are different. Because the first individual has the underwriting done more recently, the effect of underwriting on him/her is stronger, which means he/she should have a lower death probability than the second individual.

We may measure the effect of underwriting by the index of selection, which is defined as follows:

$$I(x,k) = 1 - \frac{q_{[x]+k}}{q_{x+k}}.$$

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For example, on the basis of the preceding table, $I(41,1) = 1 - q_{[41]+1}/q_{42} = 1 - 0.07/0.08 = 0.125$. If the effect of underwriting is strong, then $q_{[x]+k}$ would be small compared to q_{x+k}, and therefore $I(x,k)$ would be close to one. By contrast, if the effect of underwriting is weak, then $q_{[x]+k}$ would be close to q_{x+k}, and therefore $I(x,k)$ would be close to zero.

Let us first go through the following example, which involves a table of $q_{[x]}$.

Example 2.8. **[Course 3 Fall 2001 #2]**

For a select-and-ultimate mortality table with a 3-year select period:

(i)

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(ii) White was a newly selected life on 01/01/2000.

(iii) White's age on 01/01/2001 is 61.

(iv) P is the probability on 01/01/2001 that White will be alive on 01/01/2006.

Calculate P.

(A) $0 \leq P < 0.43$

(B) $0.43 \leq P < 0.45$

(C) $0.45 \leq P < 0.47$

(D) $0.47 \leq P < 0.49$

(E) $0.49 \leq P < 1.00$

Solution: White is now age 61 and was selected at age 60. So the probability that White will be alive 5 years from now can be expressed as $P = {}_5p_{[60]+1}$. We have

$$\begin{aligned} P &= {}_5p_{[60]+1} \\ &= p_{[60]+1} \times p_{[60]+2} \times p_{[60]+3} \times p_{[60]+4} \times p_{[60]+5} \\ &= p_{[60]+1} \times p_{[60]+2} \times p_{63} \times p_{64} \times p_{65} \\ &= (1 - q_{[60]+1})(1 - q_{[60]+2})(1 - q_{63})(1 - q_{64})(1 - q_{65}) \\ &= (1 - 0.11)(1 - 0.13)(1 - 0.15)(1 - 0.16)(1 - 0.17) \\ &= 0.4589. \end{aligned}$$

Hence, the answer is (C).

□

In some exam questions, a select-and-ultimate table may be used to model a real life problem. Take a look at the following example.

Example 2.9. **[MLC Spring 2012 #13]**

Lorie's Lorries rents lavender limousines.

On January 1 of each year they purchase 30 limousines for their existing fleet; of these, 20 are new and 10 are one-year old.

Vehicles are retired according to the following 2-year select-and-ultimate table, where selection is age at purchase:

Limousine age (x)	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
0	0.100	0.167	0.333	2
1	0.100	0.333	0.500	3
2	0.150	0.400	1.000	4
3	0.250	0.750	1.000	5
4	0.500	1.000	1.000	6
5	1.000	1.000	1.000	7

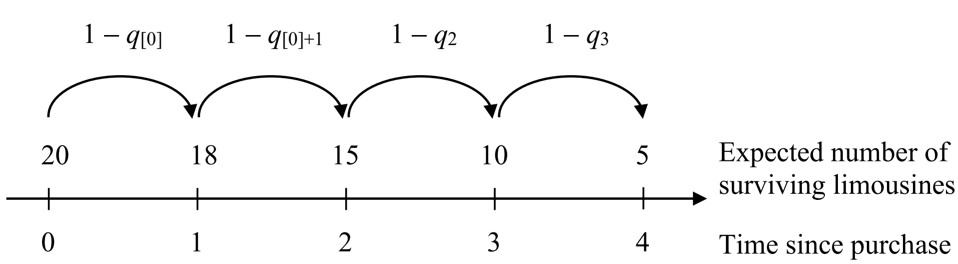
Lorie's Lorries has rented lavender limousines for the past ten years and has always purchased its limousines on the above schedule.

Calculate the expected number of limousines in the Lorie's Lorries fleet immediately after the purchase of this year's limousines.

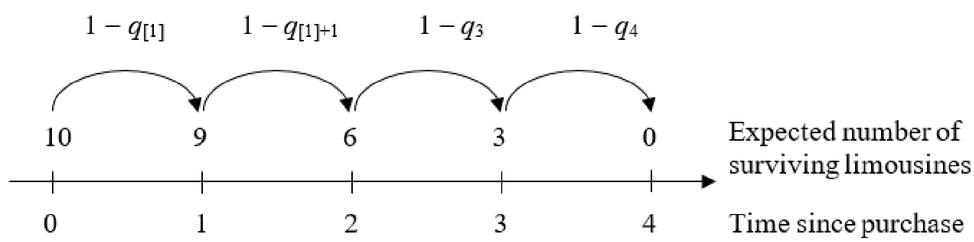
(A) 93 (B) 94 (C) 95 (D) 96 (E) 97

Solution: Let us consider a purchase of 30 limousines in a given year. According to information given, 20 of them are brand new while 10 of them are 1-year-old.

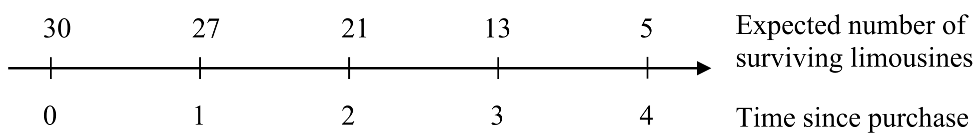
For the 20 brand new limousines, their "age at selection" is 0. As such, the sequence of "death" probabilities applicable to these 20 new limousines are $q_{[0]}$, $q_{[0]+1}$, q_2, q_3, q_4, q_5, ... Note that $q_4 = q_5 = \ldots = 1$, which implies that these limousines can last for at most four years since the time of purchase. For these 20 brand new limousines, the expected number of "survivors" limousines in each future year can be calculated as follows:



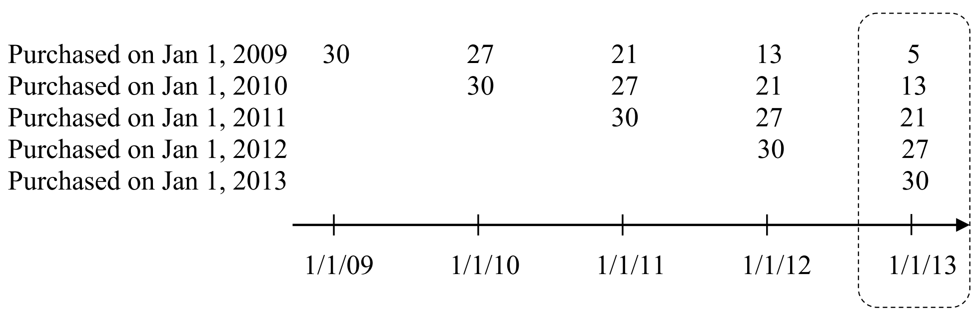
For the 10 1-year-old limousines, their "age at selection" is 1. As such, the sequence of "death" probabilities applicable to these 10 1-year-old limousines are $q_{[1]}, q_{[1]+1}, q_3, q_4, \ldots$. Note that $q_4 = q_5 = \ldots = 1$, which implies that these limousines can last for at most three years since the time of purchase. For these 10 1-year-old limousines, the expected number of "surviving" limousines in each future year can be calculated as follows:



Considering the entire purchase of 30 limousines, we have the following:



Suppose that today is January 1, 2013. Since a limousine cannot last for more than four years since the time of purchase, the oldest limousine in Lorie's fleet should be purchased on January 1, 2009. Using the results above, the expected number of limousines on January 1, 2013 can be calculated as follows:



The answer is thus $5 + 13 + 21 + 27 + 30 = 96$, which corresponds to option (D).

□

Sometimes, you may be given a select-and-ultimate table that contains the life table function l_x. In this case, you can calculate survival and death probabilities by using the following equations:

$$ {}_s p_{[x]+t} = \frac{l_{[x]+t+s}}{l_{[x]+t}}, \quad {}_s q_{[x]+t} = 1 - \frac{l_{[x]+t+s}}{l_{[x]+t}}. $$

Let us study the following two examples.

Example 2.10. You are given the following select-and-ultimate table with a 2-year select period:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
30	9907	9905	9901	32
31	9903	9900	9897	33
32	9899	9896	9892	34
33	9894	9891	9887	35

Calculate the following:

(a) ${}_2q_{[31]}$

(b) ${}_2p_{[30]+1}$

(c) ${}_{1|2}q_{[31]+1}$

Solution:

(a) $${}_2q_{[31]} = 1 - \frac{l_{[31]+2}}{l_{[31]}} = 1 - \frac{l_{33}}{l_{[31]}} = 1 - \frac{9897}{9903} = 0.000606.$$

(b) $${}_2p_{[30]+1} = \frac{l_{[30]+1+2}}{l_{[30]+1}} = \frac{l_{33}}{l_{[30]+1}} = \frac{9897}{9905} = 0.999192.$$

(c) $${}_{1|2}q_{[31]+1} = \frac{l_{[31]+1+1} - l_{[31]+1+1+2}}{l_{[31]+1}} = \frac{l_{33} - l_{35}}{l_{[31]+1}} = \frac{9897 - 9887}{9900} = 0.001010.$$ □

Exam questions such as the following may involve both $q_{[x]}$ and $l_{[x]}$.

Example 2.11. **[MLC Spring 2012 #1]**

For a 2-year select and ultimate mortality model, you are given:

(i) $q_{[x]+1} = 0.95q_{x+1}$

(ii) $l_{76} = 98{,}153$

(iii) $l_{77} = 96{,}124$

Calculate $l_{[75]+1}$.

(A) 96,150 (B) 96,780 (C) 97,420 (D) 98,050 (E) 98,690

Solution: From (ii) and (iii), we know that $q_{76} = 1 - 96124/98153 = 0.020672$.

From (i), we know that $q_{[75]+1} = 0.95q_{76} = 0.95 \times 0.020672 = 0.019638$.

Since

$$l_{[75]+2} = l_{[75]+1}(1 - q_{[75]+1}),$$

and $l_{[75]+2} = l_{77}$, we have $l_{[75]+1} = 96124/(1 - 0.019638) = 98049.5$. The answer is (D). □

It is also possible to set questions to examine your knowledge on select-and-ultimate tables and fractional age assumptions at the same time. The next example involves a select-and-ultimate table and the UDD assumption.

Example 2.12. **[Course 3 Fall 2000 #10]**

You are given the following extract from a select-and-ultimate mortality table with a 2-year select period:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
60	80625	79954	78839	62
61	79137	78402	77252	63
62	77575	76770	75578	64

Assume that deaths are uniformly distributed between integral ages.

Calculate ${}_{0.9}q_{[60]+0.6}$.

(A) 0.0102 (B) 0.0103 (C) 0.0104 (D) 0.0105 (E) 0.0106

Solution: We illustrate two methods:

(1) Interpolation

The live age $q_{[60]+0.6}$ is originally selected at age [60]. So, we can use $l_{[60]} = 80625, l_{[60]+1} = 79954, l_{[60]+2} = l_{62} = 78839$ and so on to calculate mortality rate.

$$ {}_{0.9}q_{[60]+0.6} = 1 - \frac{l_{[60]+1.5}}{l_{[60]+0.6}} $$

$$ \begin{aligned} l_{[60]+0.6} &= 0.4l_{[60]} + 0.6l_{[60]+1} \\ &= 0.4 \times 80625 + 0.6 \times 79954 \\ &= 80222.4 \end{aligned} $$

$$ \begin{aligned} l_{[60]+1.5} &= 0.5l_{[60]+1} + 0.5l_{[60]+2} \\ &= 0.5 \times 79954 + 0.6 \times 78839 \\ &= 79396.5 \end{aligned} $$

The death probability is $1 - \dfrac{79396.5}{80222.4} = 0.010295$.

(2) The trick we have introduced to shift the fractional age to integral age

Recall that when UDD is assumed and the subscript on the right-hand-side is not an integer, we will need to use the trick. We first calculate ${}_{0.9}p_{[60]+0.6}$. Using the trick, we have

$$ {}_{0.6}p_{[60]} \times {}_{0.9}p_{[60]+0.6} = {}_{1.5}p_{[60]}. $$

Then, we have

$$
\begin{aligned}
{}_{0.9}p_{[60]+0.6} &= \frac{{}_{1.5}p_{[60]}}{{}_{0.6}p_{[60]}} = \frac{p_{[60]}\ {}_{0.5}p_{[60]+1}}{{}_{0.6}p_{[60]}} \\
&= \frac{p_{[60]}(1-0.5q_{[60]+1})}{1-0.6q_{[60]}} \\
&= \frac{\dfrac{l_{[60]+1}}{l_{[60]}}\left[1-0.5\left(1-\dfrac{l_{[60]+2}}{l_{[60]+1}}\right)\right]}{1-0.6\left(1-\dfrac{l_{[60]+1}}{l_{[60]}}\right)} \\
&= \frac{\dfrac{79954}{80625}\left[1-0.5\left(1-\dfrac{78839}{79954}\right)\right]}{1-0.6\left(1-\dfrac{79954}{80625}\right)} \\
&= 0.989705.
\end{aligned}
$$

As a result, ${}_{0.9}q_{[60]+0.6} = 1 - 0.989705 = 0.0103$. Hence, the answer is (B).

□

The following example involves a select-and-ultimate table and the constant force of mortality assumption.

Example 2.13. **[MLC Fall 2012 #2]**

You are given:

(i) An excerpt from a select and ultimate life table with a select period of 3 years.

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}	$x+3$
60	80,000	79,000	77,000	74,000	63
61	78,000	76,000	73,000	70,000	64
62	75,000	72,000	69,000	67,000	65
63	71,000	68,000	66,000	65,000	66

(ii) Deaths follow a constant force of mortality over each year of age.

Calculate $1000\ {}_{2|3}q_{[60]+0.75}$.

(A) 104 (B) 117 (C) 122 (D) 135 (E) 142

Solution: As discussed in Section 2.2, there are two methods for solving such a problem.

Method 1: Interpolation

The probability required is ${}_{2|3}q_{[60]+0.75} = \dfrac{l_{[60]+2.75} - l_{[60]+5.75}}{l_{[60]+0.75}} = \dfrac{l_{[60]+2.75} - l_{65.75}}{l_{[60]+0.75}}$.

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Under the constant force of mortality assumption, we have

$$\begin{aligned}\ln l_{[60]+0.75} &= 0.25 \ln l_{[60]} + 0.75 \ln l_{[60]+1} = 0.25 \ln 80000 + 0.75 \ln 79000 \\ &\Rightarrow l_{[60]+0.75} = \exp(11.28035) = 79248.82\end{aligned}$$

$$\begin{aligned}\ln l_{[60]+2.75} &= 0.25 \ln l_{[60]+2} + 0.75 \ln l_{63} = 0.25 \ln 77000 + 0.75 \ln 74000 \\ &\Rightarrow l_{[60]+2.75} = \exp(11.22176) = 74738.86\end{aligned}$$

$$\begin{aligned}\ln l_{65.75} &= 0.25 \ln l_{65} + 0.75 \ln l_{66} = 0.25 \ln 67000 + 0.75 \ln 65000 \\ &\Rightarrow l_{63.75} = \exp(11.08972) = 65494.33\end{aligned}$$

As a result, ${}_{2|3}q_{[60]+0.75} = (74738.86 - 65494.33)/79248.82 = 0.11665$.

<u>Method 2</u>: Working on the survival probabilities

The probability required is

$$\begin{aligned}{}_{2|3}q_{[60]+0.75} &= {}_2p_{[60]+0.75} - {}_5p_{[60]+0.75} \\ &= {}_{0.25}p_{[60]+0.75} \times p_{[60]+1} \times {}_{0.75}p_{[60]+2} - {}_{0.25}p_{[60]+0.75} \times {}_4p_{[60]+1} \times {}_{0.75}p_{[60]+5} \\ &= \left(\frac{79}{80}\right)^{1/4} \times \frac{77}{79} \times \left(\frac{74}{77}\right)^{0.75} - \left(\frac{79}{80}\right)^{1/4} \times \frac{67}{79} \times \left(\frac{65}{67}\right)^{0.75} \\ &= 0.11665\end{aligned}$$

Both methods imply $1000 \, {}_{2|3}q_{[60]+0.75} = 116.65$, which corresponds to option (B).

□

2.4 Moments of Future Lifetime Random Variables

In Exam P, you learnt how to calculate the moments of a random variable.

- If W is a discrete random variable, then $\mathrm{E}(W) = \sum_w w\Pr(W = w)$.
- If W is a continuous random variable, then $\mathrm{E}(W) = \int_{-\infty}^{\infty} w f_W(w)\mathrm{d}w$, where $f_W(w)$ is the density function for W.
- To calculate variance, we can always use the identity $\mathrm{Var}(W) = \mathrm{E}(W^2) - [\mathrm{E}(W)]^2$.

First, let us focus on the moments of the future lifetime random variable T_x. We call $\mathrm{E}(T_x)$ the complete expectation of life at age x, and denote it by $\mathring{e}_x$. We have

$$\mathring{e}_x = \int_0^{\infty} t f_x(t)\mathrm{d}t = \int_0^{\infty} t\,{}_tp_x \mu_{x+t}\mathrm{d}t.$$

By rewriting the integral as $-\int_0^{\infty} t\mathrm{d}S_x(t)$ and using integration by parts, we have

$$\mathring{e}_x = -[tS_x(t)]_0^{\infty} + \int_0^{\infty} S_x(t)\mathrm{d}t = \int_0^{\infty} {}_tp_x\mathrm{d}t.$$

Note that if there is a limiting age, we replace ∞ with $\omega - x$.

The second moment of T_x can be expressed as

$$\mathrm{E}(T_x^2) = \int_0^{\infty} t^2 f_x(t)\mathrm{d}t.$$

Using integration by parts, we can show that the above formula can be rewritten as

$$\mathrm{E}(T_x^2) = 2\int_0^{\infty} t\ {}_tp_x\mathrm{d}t,$$

which is generally easier to apply. Again, if there is a limiting age, we replace ∞ with $\omega - x$.

In the exam, you may also be asked to calculate $\mathrm{E}(T_x \wedge n) = \mathrm{E}[\min(T_x, n)]$. This expectation is known as the n-year temporary complete expectation of life at age x, and is denoted by $\mathring{e}_{x:\overline{n}|}$. By definition,

$$\mathring{e}_{x:\overline{n}|} = \int_0^{n} t f_x(t)\mathrm{d}t + \int_n^{\infty} n f_x(t)\mathrm{d}t = -\int_0^{n} t\mathrm{d}S_x(t) + n\,{}_np_x.$$

Then it follows again from integration by parts that

$$\mathring{e}_{x:\overline{n}|} = \int_0^{n} {}_tp_x\mathrm{d}t.$$

The following is a summary of the formulas for the moments of T_x.

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Moments of T_x

(2.7) $$\mathring{e}_x = \int_0^\infty {}_tp_x \mathrm{d}t$$

(2.8) $$\mathrm{E}(T_x^2) = 2\int_0^\infty t\ {}_tp_x \mathrm{d}t$$

(2.9) $$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \mathrm{d}t$$

Example 2.14.

You are given $\mu_x = 0.01$ for all $x \geq 0$. Calculate the following:

(a) $\mathring{e}_x$

(b) $\mathrm{Var}(T_x)$

Solution:

(a) First of all, we have ${}_tp_x = e^{-0.01t}$. Then,

$$\mathring{e}_x = \int_0^\infty {}_tp_x\ \mathrm{d}t = \int_0^\infty e^{-0.01t}\ \mathrm{d}t = \frac{1}{-0.01}\left[e^{-0.01t}\right]_0^\infty = \frac{1}{0.01} = 100.$$

(b) We first calculate the second moment of T_x as follows:

$$\begin{aligned}\mathrm{E}(T_x^2) &= 2\int_0^\infty te^{-0.01t}\ \mathrm{d}t \\ &= \frac{-2}{0.01}\left(te^{-0.01t}\Big|_0^\infty - \int_0^\infty e^{-0.01t}\ \mathrm{d}t\right) \\ &= \frac{2}{0.01}\int_0^\infty e^{-0.01t}\ \mathrm{d}t \\ &= \frac{2}{0.01^2} = 20000.\end{aligned}$$

Then, the variance of T_x can be calculated as:

$$\begin{aligned}\mathrm{Var}(T_x) &= \mathrm{E}(T_x^2) - \left[\mathrm{E}(T_x)\right]^2 \\ &= 20000 - 100^2 \\ &= 10000.\end{aligned}$$

Alternatively, from $S_x(t) = e^{-0.01t}$, we see that $F_x(t) = 1 - e^{-0.01t}$. Since this is the cumulative distribution function of an exponential random variable with rate 0.01, it is immediate that T_x is exponentially distributed with rate 0.01. Hence, the mean is $1/0.01 = 100$ and the variance is $1/0.01^2 = 10000$. □

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Example 2.15. **[Course 3 Fall 2001 #1]**

You are given:

$$\mu_x = \begin{cases} 0.04, & 0 < x < 40 \\ 0.05, & x > 40 \end{cases}$$

Calculate $\mathring{e}_{25:\overline{25}|}$.

(A) 14.0 (B) 14.4 (C) 14.8 (D) 15.2 (E) 15.6

Solution: First, we need to find ${}_tp_x$. Because the value of μ_x changes when x reaches 40, the derivation of ${}_tp_x$ is not as straightforward as that in the previous example.

For $0 < t < 15$, μ_{25+t} is always 0.04, and therefore

$${}_tp_{25} = \exp\left(-\int_0^t 0.04\text{d}u\right) = e^{-0.04t}.$$

For $t > 15$, μ_{25+t} becomes 0.05, and therefore

$${}_tp_{25} = {}_{15}p_{25} \times {}_{t-15}p_{40} = e^{-0.04\times 15}\exp\left(-\int_0^{t-15} 0.05\text{d}u\right) = e^{-0.04\times 15 - 0.05(t-15)} = e^{-0.05t+0.15}.$$

Given the expressions for ${}_tp_{25}$, we can calculate $\mathring{e}_{25:\overline{25}|}$ as follows:

$$\begin{aligned}\mathring{e}_{25:\overline{25}|} &= \int_0^{15} {}_tp_{25}\ \text{d}t + \int_{15}^{25} {}_tp_{25}\ \text{d}t \\ &= \int_0^{15} e^{-0.04t}\ \text{d}t + \int_{15}^{25} e^{-0.05t+0.15}\ \text{d}t \\ &= \left.\frac{e^{-0.04t}}{-0.04}\right|_0^{15} + e^{0.15}\left[\frac{e^{-0.05t}}{-0.05}\right]_{15}^{25} = 15.60.\end{aligned}$$

Hence, the answer is (E). □

Example 2.16. **[Structural Question SOA Sample #1]**

You are given the following survival function for a newborn:

$$S_0(t) = \frac{(121-t)^{1/2}}{k}, \quad 0 \le t \le \omega.$$

(a) Show that k must be 11 for $S_0(t)$ to be a valid survival function.

(b) Show that the limiting age, ω, for this survival model is 121.

(c) Calculate $\mathring{e}_0$ for this survival model.

(d) Derive an expression for μ_x for this survival model, simplifying the expression as much as possible.

(e) Calculate the probability, using the above survival model, that (57) dies between the ages of 84 and 100.

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Solution:

(a) Recall that the first criterion for a valid survival function is that $S_0(0) = 1$. This implies that

$$\begin{aligned}\frac{(121-0)^{1/2}}{k} &= 1\\ (121)^{1/2} &= k\\ k &= 11\end{aligned}$$

(b) At the limiting age, the value of the survival function must be zero. Therefore,

$$\begin{aligned}S_0(\omega) &= 0\\ \frac{(121-\omega)^{1/2}}{k} &= 0\\ \omega &= 121\end{aligned}$$

(c) Using formula (2.8) with $x = 0$, we have

$$\begin{aligned}\mathring{e}_0 &= \int_0^{\omega-0} {}_tp_0 \; \mathrm{d}t\\ &= \int_0^{121} \frac{(121-t)^{1/2}}{11} \; \mathrm{d}t\\ &= \frac{1}{11}\left[\frac{-2}{3}(121-t)^{3/2}\right]_0^{121}\\ &= 80.6667\end{aligned}$$

(d) This part involves the relationship between the μ_x and $S_0(x)$, which was taught in Chapter 1:

$$\begin{aligned}\mu_x = -\frac{S_0'(x)}{S_0(x)} &= -\frac{\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{(121-x)^{1/2}}{k}}{\dfrac{(121-x)^{1/2}}{k}}\\ &= -\frac{\dfrac{\mathrm{d}}{\mathrm{d}x}(121-x)^{1/2}}{(121-x)^{1/2}} = \frac{\dfrac{1}{2}(121-x)^{-1/2}}{(121-x)^{1/2}} = \frac{1}{2(121-x)}\end{aligned}$$

(e) First, we derive an expression for $S_{57}(t)$ as follows:

$$S_{57}(t) = \frac{S_0(57+t)}{S_0(57)} = \frac{\dfrac{(121-(57+t))^{1/2}}{11}}{\dfrac{(121-57)^{1/2}}{11}} = \sqrt{\frac{64-t}{64}}$$

The required probability is ${}_{27|16}q_{57}$, which can be calculated as follows:

$${}_{27|16}q_{57} = S_{57}(27) - S_{57}(43) = \sqrt{\frac{64-27}{64}} - \sqrt{\frac{64-43}{64}} = 0.1875$$

□

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Now, we focus on the moments of the curtate future lifetime random variable K_x. The first moment of K_x is called the curtate expectation of life at age x, and is denoted by e_x. The formula for calculating e_x is derived as follows:

$$\begin{aligned}
e_x &= \mathrm{E}(K_x) \\
&= \sum_{k=0}^{\infty} k\Pr(K_x = k) = \sum_{k=0}^{\infty} k\ {}_{k|}q_x \\
&= 0 \times q_x + 1 \times {}_{1|1}q_x + 2 \times {}_{2|1}q_x + 3 \times {}_{3|1}q_x + \ldots \\
&= (p_x - {}_2p_x) + 2\,({}_2p_x - {}_3p_x) + 3\,({}_3p_x - {}_4p_x) + \ldots \\
&= p_x + {}_2p_x + {}_3p_x + \ldots \\
&= \sum_{k=1}^{\infty} {}_kp_x.
\end{aligned}$$

If there is a limiting age, we replace ∞ with $\omega - x$.

The formula for calculating the second moment of K_x can be derived as follows:

$$\begin{aligned}
\mathrm{E}(K_x^2) &= \sum_{k=0}^{\infty} k^2\Pr(K_x = k) = \sum_{k=0}^{\infty} k^2\ {}_{k|}q_x \\
&= 0^2 \times q_x + 1^2 \times {}_{1|1}q_x + 2^2 \times {}_{2|1}q_x + 3^2 \times {}_{3|1}q_x + \ldots \\
&= (p_x - {}_2p_x) + 4\,({}_2p_x - {}_3p_x) + 9\,({}_3p_x - {}_4p_x) + \ldots \\
&= p_x + 3\ {}_2p_x + 5\ {}_3p_x + \ldots \\
&= \sum_{k=1}^{\infty} (2k-1)_kp_x.
\end{aligned}$$

Again, if there is a limiting age, we replace ∞ with $\omega - x$. Given the two formulas above, we can easily obtain $\mathrm{Var}(K_x)$.

In the exam, you may also be asked to calculate $\mathrm{E}(K_x \wedge n) = \mathrm{E}[\min(K_x, n)]$. This is called the n-year temporary curtate expectation of life at age x, and is denoted by $e_{x:\overline{n}|}$. It can be shown that

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x,$$

that is, instead of summing to infinity, we just sum to n.

There are two other equations that you need to know. First, you need to know that e_x and e_{x+1} are related to each other as follows:

$$e_x = p_x(1 + e_{x+1}).$$

Formulas of this form are called recursion formulas. We will further discuss recursion formulas in Chapters 3 and 4.

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Second, assuming UDD holds, we have $T_x = K_x + U$, where U follows a uniform distribution over the interval [0,1]. Taking expectation on both sides, we have the following relation:

$$\mathring{e}_x = e_x + \frac{1}{2}.$$

The following is a summary of the key equations for the moments of K_x.

Moments of K_x

(2.10) $$e_x = \sum_{k=1}^{\infty} {}_k p_x$$

(2.11) $$\mathrm{E}(K_x^2) = \sum_{k=1}^{\infty} (2k-1)\,{}_k p_x$$

(2.12) $$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_k p_x$$

(2.13) $$e_x = p_x(1 + e_{x+1})$$

(2.14) $$\text{Under UDD, } \mathring{e}_x = e_x + \frac{1}{2}$$

Example 2.17. You are given the following excerpt of a life table:

x	95	96	97	98
l_x	400	300	100	0

Calculate the following:

(a) e_{95}

(b) $\mathrm{Var}(K_{95})$

(c) $e_{95:\overline{1}|}$

(d) $\mathring{e}_{95}$, assuming UDD

(e) e_{96}, using the recursion formula

Solution:

(a) $e_{95} = \sum_{k=1}^{3} {}_k p_{95} = \frac{l_{96}}{l_{95}} + \frac{l_{97}}{l_{95}} + \frac{l_{98}}{l_{95}} = \frac{300}{400} + \frac{100}{400} + \frac{0}{400} = 1$

(b) We have

$$\mathrm{E}(K_{95}^2) = \sum_{k=1}^{3} (2k-1)\,{}_k p_{95} = \frac{l_{96}}{l_{95}} + \frac{3l_{97}}{l_{95}} + \frac{5l_{98}}{l_{95}} = \frac{300 + 3 \times 100 + 5 \times 0}{400} = 1.5.$$

Hence, $\mathrm{Var}(K_{95}) = 1.5 - 1^2 = 0.5$.

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(c) $e_{95:\overline{1}|} = \sum_{k=1}^{1} {}_kp_{95} = \frac{l_{96}}{l_{95}} = \frac{300}{400} = 0.75.$

(d) Assuming UDD, $\mathring{e}_{95} = e_{95} + 0.5 = 1 + 0.5 = 1.5.$

(e) Using the recursion formula, $e_{95} = p_{95}(1 + e_{96}) = 0.75(1 + e_{96})$. Therefore,

$$e_{96} = 1/0.75 - 1 = 0.3333.$$

□

Example 2.18. **[MLC Fall 2012 #3]**

You are given:

(i) $S_0(t) = \left(1 - \frac{t}{\omega}\right)^{1/4}$, for $0 \le t \le \omega$

(ii) $\mu_{65} = 1/180$

Calculate e_{106}, the curtate expectation of life at age 106.

(A) 2.2 (B) 2.5 (C) 2.7 (D) 3.0 (E) 3.2

Solution: From statement (i), we know that there is a limiting age ω. Our first step is to compute the value of ω, using the information given.

Since

$$\mu_x = -\frac{\mathrm{d}}{\mathrm{d}x} \ln S_0(x) = \frac{1}{4(\omega - x)},$$

by statement (ii) we have

$$\frac{1}{4(\omega - 65)} = \frac{1}{180},$$

or

$$\omega = 110.$$

Then, using formula (2.9), we can calculate e_{106} as follows:

$$\begin{aligned} e_{106} &= p_{106} + {}_2p_{106} + {}_3p_{106} + {}_4p_{106} + \dots \\ &= \frac{S_0(107) + S_0(108) + S_0(109) + S_0(110) + \dots}{S_0(106)} \\ &= \frac{0.02727^{1/4} + 0.01818^{1/4} + 0.00090^{1/4} + 0 + \dots}{0.03636^{1/4}} \\ &= 2.4786 \end{aligned}$$

The answer is (B).

□

Example 2.19. **[MLC Spring 2012 #2]**

You are given:

(i) $p_x = 0.97$

(ii) $p_{x+1} = 0.95$

(iii) $e_{x+1.75} = 18.5$

(iv) Deaths are uniformly distributed between ages x and $x+1$.

(v) The force of mortality is constant between ages $x+1$ and $x+2$.

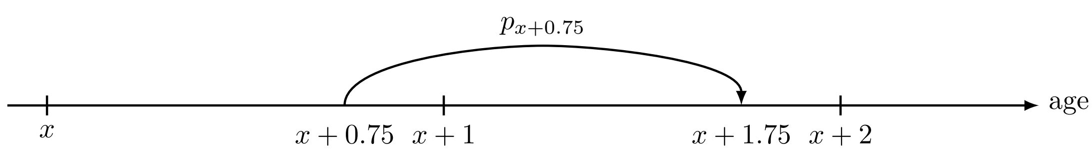
Calculate $e_{x+0.75}$.

(A) 18.6 (B) 18.8 (C) 19.0 (D) 19.2 (E) 19.4

Solution: Our goal is to calculate $e_{x+0.75}$. Since we are given the value of $e_{x+1.75}$, it is quite obvious that we should use the following recursive relation:

$$e_{x+0.75} = p_{x+0.75}\left(1 + e_{x+1.75}\right).$$

All then that remains is to calculate $p_{x+0.75.}$ As shown in the following diagram, this survival probability covers part of the interval $[x, x+1)$ and part of the interval $[x+1, x+2)$.



We shall apply fractional age assumptions accordingly. Decomposing $p_{x+0.75}$, we have

$$p_{x+0.75} = {}_{0.25}p_{x+0.75} \times {}_{0.75}p_{x+1}.$$

According to statement (iv), the value of ${}_{0.25}p_{x+0.75}$ should be calculated by assuming UDD. Under this assumption, we have

$${}_{0.25}p_{x+0.75} = \frac{p_x}{{}_{0.75}p_x} = \frac{0.97}{1 - 0.75(1-0.97)} = 0.992327366.$$

According to statement (v), the value of ${}_{0.75}p_{x+1}$ should be calculated by assuming constant force of mortality over each year of age. Under this assumption, we have

$${}_{0.75}p_{x+1} = (p_{x+1})^{0.75} = (0.95)^{0.75} = 0.9622606.$$

It follows that $p_{x+0.75} = 0.992327366 \times 0.9622606 = 0.954878$.

Finally,

$$e_{x+0.75} = 0.954878 \times (1 + 18.5) = 18.620.$$

The answer is (A).

□

2.5 Useful Shortcuts

Constant Force of Mortality for All Ages

Very often, you are given that $\mu_x = \mu$ for all $x \geq 0$. In this case, we can easily find that

$$ {}_tp_x = e^{-\mu t}, \quad F_x(t) = 1 - e^{-\mu t}, \quad f_x(t) = \mu e^{-\mu t}. $$

From the density function, you can tell that in this case T_x follows an exponential distribution with parameter μ. By using the properties of an exponential distribution, we have

$$ \mathring{e}_x = \mathrm{E}(T_x) = 1/\mu, \quad \mathrm{Var}(T_x) = 1/\mu^2 \quad \text{for all } x. $$

These shortcuts can save you a lot of time on doing integration. For instance, had you known these shortcuts, you could complete Example 2.14 in a blink!

De Moivre's Law

De Moivre's law refers to the situation when

$$ l_x = \omega - x \text{ for } 0 \leq x < \omega, $$

or equivalently

$$ \mu_x = \frac{1}{\omega - x}. $$

De Moivre's law implies that the age at death random variable (T_0) is uniformly distributed over the interval $[0, \omega)$. It also implies that the future lifetime random variable (T_x) is uniformly distributed over the interval $[0, \omega - x)$, that is, for $0 \leq t < \omega - x$,

$$ {}_tp_x = 1 - \frac{t}{\omega - x}, \quad F_x(t) = \frac{t}{\omega - x}, \quad f_x(t) = \frac{1}{\omega - x}, \quad \mu_{x+t} = \frac{1}{\omega - x - t}. $$

By using the properties of uniform distributions, we can immediately obtain

$$ \mathring{e}_x = \frac{\omega - x}{2}, \quad \mathrm{Var}(T_x) = \frac{(\omega - x)^2}{12}. $$

The useful shortcuts are summarized in the following table.

Assumption	μ_{x+t}	${}_tp_x$	$F_x(t)$	$f_x(t)$	$\mathring{e}_x$	$\mathrm{Var}(T_x)$
Constant force for all ages	μ	$e^{-\mu t}$	$1 - e^{-\mu t}$	$\mu e^{-\mu t}$	$1/\mu$	$1/\mu^2$
De Moivre's law	$\frac{1}{\omega - x - t}$	$1 - \frac{t}{\omega - x}$	$\frac{t}{\omega - x}$	$\frac{1}{\omega - x}$	$\frac{\omega - x}{2}$	$\frac{(\omega - x)^2}{12}$

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Example 2.20. You are given:

$$l_x = 100 - x, \quad 0 \le x < 100.$$

Calculate the following:

(a) ${}_{25}p_{25}$

(b) q_{25}

(c) μ_{50}

(d) $\mathring{e}_{50}$

Solution: First of all, note that $l_x = 100 - x$ for $0 \le x < 100$ means mortality follows De Moivre's law with $\omega = 100$.

(a) ${}_{25}p_{25} = 1 - \dfrac{25}{100-25} = \dfrac{2}{3}$.

(b) $q_{25} = 1 - l_{26}/l_{25} = 1 - 74/75 = 1/75$.
Alternatively, you can obtain the answer by using the fact that T_{25} is uniformly distributed over the interval [0,75). It immediately follows that the probability that (25) dies within one year is 1/75.

(c) $\mu_{50} = \dfrac{1}{100-50} = 0.02$.

(d) $\mathring{e}_{50} = \dfrac{100-50}{2} = 25$.

□

Example 2.21.

The survival function for the age-at-death random variable is given by

$$S_0(t) = 1 - \frac{t}{\omega}, \quad t \le \omega.$$

(a) Find an expression for $S_x(t)$, for $x < \omega$ and $t \le \omega - x$.

(b) Show that $\mu_x = \dfrac{1}{\omega - x}$, for $x < \omega$.

(c) Assuming $\mathring{e}_0 = 25$, show that $\omega = 50$.

Solution:

(a) The survival function implies that the age-at-death random variable is uniformly distributed over $[0, \omega]$. De Moivre's law applies here, so we can immediately write down the expression for $S_x(t)$ as follows:

$$S_x(t) = 1 - \frac{t}{\omega - x}.$$

(b) From Section 2.5, we know the expression for μ_x. However, since we are asked to prove the relation, we should show the steps involved:

$$\mu_x = -\frac{S_0'(x)}{S_0(x)} = -\frac{\frac{-1}{\omega}}{1-\frac{x}{\omega}} = \frac{1}{\omega - x}$$

(c) Under De Moivre's law, $\mathring{e}_0 = \frac{\omega}{2}$. Hence, we have $\frac{\omega}{2} = 25$, which gives $\omega = 50$.

□

Example 2.22.

It is given that $\mu_x = \mu$ for all $x \geq 0$.

(a) Show that $\mathring{e}_{x:\overline{n}|} = \dfrac{1-e^{-\mu n}}{\mu}$.

(b) Explain verbally why $\mathring{e}_{x:\overline{n}|}$ does not depend on x when we assume $\mu_x = \mu$ for all $x \geq 0$.

(c) State the value of $\mathring{e}_{x:\overline{n}|}$ when μ tends to zero. Explain your answer.

Solution:

(a) Since the force of mortality is constant for all ages, we have ${}_tp_x = e^{-\mu t}$. Then,

$$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \mathrm{d}t = \int_0^n e^{-\mu t}\mathrm{d}t = \frac{1-e^{-\mu n}}{\mu}.$$

(b) The assumption "$\mu_x = \mu$ for all $x \geq 0$" means that the future lifetime random variable is **exponentially distributed**. By the **memoryless property** of an exponential distribution, the expectation should be independent of the history (i.e., how long the life has survived).

(c) When μ tends to zero, $\mathring{e}_{x:\overline{n}|}$ tends to n. This is because when μ tends to zero, the underlying lives become immortal (i.e., the lives will live forever). As a result, the average number of years survived from age x to age $x+n$ (i.e., from time 0 to time n) must be n.

□

2.6 Exercise 2

1. You are given the following excerpt of a life table:

x	l_x
50	100,000
51	99,900
52	99,700
53	99,500
54	99,100
55	98,500

Calculate the following:

(a) ${}_2d_{52}$

(b) ${}_{3|}q_{50}$

2. You are given:

$$l_x = 10000e^{-0.05x}, \quad x \geq 0.$$

Find ${}_{5|15}q_{10}$.

3. You are given the following excerpt of a life table:

x	l_x
40	10,000
41	9,900
42	9,700
43	9,400
44	9,000
45	8,500

Assuming uniform distribution of deaths between integral ages, calculate the following:

(a) ${}_{0.2}p_{42}$

(b) ${}_{2.6}q_{41}$

(c) ${}_{1.6}q_{40.9}$

4. Repeat Question 3 by assuming constant force of mortality between integral ages.

5. You are given:

(i) $l_{40} = 9{,}313{,}166$

(ii) $l_{41} = 9{,}287{,}264$

(iii) $l_{42} = 9{,}259{,}571$

Assuming uniform distribution of deaths between integral ages, find ${}_{1.4}q_{40.3}$.

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6. You are given:

x	l_x
40	60500
50	55800
60	50200
70	44000
80	36700

Assuming that deaths are uniformly distributed over each 10-year interval, find ${}_{15|20}q_{40}$.

7. You are given the following select-and-ultimate table with a select period of 2 years:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
50	0.02	0.04	0.06	52
51	0.03	0.05	0.07	53
52	0.04	0.06	0.08	54

Find ${}_{2|2}q_{[50]}$.

8. You are given the following select-and-ultimate table with a select period of 2 years:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
70	22507	22200	21722	72
71	21500	21188	20696	73
72	20443	20126	19624	74
73	19339	19019	18508	75
74	18192	17871	17355	76

(a) Compute ${}_3p_{73}$.

(b) Compute the probability that a life age 71 dies between ages 75 and 76, given that the life was selected at age 70.

(c) Assuming uniform distribution of deaths between integral ages, calculate ${}_{0.5}p_{[70]+0.7}$.

(d) Assuming constant force of mortality between integral ages, calculate ${}_{0.5}p_{[70]+0.7}$.

9. You are given:

$$f_0(t) = \frac{20-t}{200}, \quad 0 \le t < 20.$$

Find $\mathring{e}_5$.

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10. For a certain individual, you are given:

$$S_0(t) = \begin{cases} 1 - \dfrac{t}{100}, & 0 \le t < 30 \\ 0.7e^{-0.02(t-30)}, & t \ge 30 \end{cases}$$

Calculate $\mathrm{E}(T_0)$ for the individual.

11. You are given:

$$\mu_x = \frac{2x}{400 - x^2}, \quad 0 \le x < 20.$$

Find $\mathrm{Var}(T_0)$.

12. You are given:

(i) $\mu_x = \dfrac{1}{\omega - x}, \quad 0 \le x < \omega.$

(ii) $\mathrm{Var}(T_0) = 468.75.$

Find ω.

13. You are given:

(i) $\mu_x = \mu$ for all $x \ge 0$.

(ii) $\mathring{e}_{30} = 40.$

Find ${}_5p_{20}$.

14. You are given:

$$l_x = 10000 - x^2, \quad 0 \le x \le 100.$$

Find $\mathrm{Var}(T_0)$.

15. You are given:

$$\mu_x = 0.02, \quad x \ge 0.$$

Find $\mathring{e}_{10:\overline{10}|}$.

16. You are given:

(i) $S_0(t) = 1 - \dfrac{t}{\omega}, \quad 0 \le t < \omega.$

(ii) $\mathring{e}_{20:\overline{30}|} = 22.5.$

Calculate $\mathrm{Var}(T_{30})$.

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17. You are given:

$$l_x = 80 - x, \quad 0 \le x \le 80.$$

Find $\mathring{e}_{5:\overline{15}|}$.

18. (CAS, 2003 Fall #5) You are given:

(a) Mortality follows De Moivre's Law.

(b) $\mathring{e}_{20} = 30$.

Calculate q_{20}.

(A) 1/60 (B) 1/70 (C) 1/80 (D) 1/90 (E) 1/100

19. (2005 Nov #32) For a group of lives aged 30, containing an equal number of smokers and non-smokers, you are given:

(i) For non-smokers, $\mu_x^n = 0.08, x \ge 30$.

(ii) For smokers, $\mu_x^s = 0.16, x \ge 30$.

Calculate q_{80} for a life randomly selected from those surviving to age 80.

(A) 0.078 (B) 0.086 (C) 0.095 (D) 0.104 (E) 0.112

20. (2004 Nov #4) For a population which contains equal numbers of males and females at birth:

(i) For males: $\mu_x^m = 0.10, x \ge 0$.

(ii) For females: $\mu_x^f = 0.08, x \ge 0$.

Calculate q_{60} for this population.

(A) 0.076 (B) 0.081 (C) 0.086 (D) 0.091 (E) 0.096

21. (2000 May #1) You are given:

(i) $\mathring{e}_0 = 25$

(ii) $l_x = \omega - x, \quad 0 \le x \le \omega$.

(iii) T_x is the future lifetime random variable.

Calculate $\mathrm{Var}(T_{10})$.

(A) 65 (B) 93 (C) 133 (D) 178 (E) 333

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22. (2005 May #21) You are given:

(i) $\mathring{e}_{30:\overline{40}|} = 27.692$

(ii) $S_0(t) = 1 - t/\omega, \quad 0 \le t \le \omega.$

(iii) T_x is the future lifetime random variable for (x).

Calculate $\text{Var}(T_{30})$.

23. (2005 Nov # 13) The actuarial department for the SharpPoint Corporation models the lifetime of pencil sharpeners from purchase using a generalized De Moivre model with $S_0(t) = (1 - t/\omega)^\alpha$, for $\alpha > 0$ and $0 \le t \le \omega$.

A senior actuary examining mortality tables for pencil sharpeners has determined that the original value of α must change. You are given:

(i) The new complete expectation of life at purchase is half what it was previously.

(ii) The new force of mortality for pencil sharpeners is 2.25 times the previous force of mortality for all durations.

(iii) ω remains the same.

Calculate the original value of α.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

24. (2000 Nov #25) You are given:

(i) Superscripts M and N identify two forces of mortality and the curtate expectations of life calculated from them.

(ii) $\mu^N_{25+t} = \begin{cases} \mu^M_{25+t} + 0.10(1-t), & 0 \le t \le 1 \\ \mu^M_{25+t}, & t > 1 \end{cases}$

(iii) $e^M_{25} = 10.0$

Calculate e^N_{25}.

(A) 9.2 (B) 9.3 (C) 9.4 (D) 9.5 (E) 9.6

25. (2003 Nov # 17) T_0, the future lifetime of (0), has a spliced distribution:

(i) $f^a(t)$ follows the Illustrative Life Table.

(ii) $f^b(t)$ follows De Moivre's law with $\omega = 100$.

(iii) The density function of T_0 is $f_0(t) = \begin{cases} kf^a(t), & 0 \le t \le 50 \\ 1.2f^b(t), & t > 50 \end{cases}$

Calculate ${}_{10}p_{40}$.

(A) 0.81 (B) 0.85 (C) 0.88 (D) 0.92 (E) 0.96

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26. (a) Show that $e_x = p_x(1 + e_{x+1})$.

(b) Show that if deaths are uniformly distributed between integer ages, then

$$\mathring{e}_x = e_x + \frac{1}{2}.$$

(c) For a life table with a one-year select period, you are given:

x	$l_{[x]}$	$d_{[x]}$	l_{x+1}	$\mathring{e}_{[x]}$
80	1000	90	–	8.5
81	920	90	–	–

(i) Find l_{81} and l_{82}.

(ii) Assuming deaths are uniformly distributed over each year of age, calculate $\mathring{e}_{[81]}$.

27. For a certain group of individuals, you are given:

$$F_0(t) = 1 - e^{-0.02t}, \quad t \geq 0.$$

(a) Show that $S_x(t) = e^{-0.02t}$ for $x, t \geq 0$.

(b) Show that $\mu_x = 0.02$ for $x \geq 0$.

(c) Calculate $\mathring{e}_{10:\overline{10}|}$.

(d) Calculate e_{10}.

28. Consider the curtate future lifetime random variable, K_x.

(a) Explain verbally why $\Pr(K_x = k) = {}_{k|}q_x$ for $k = 0, 1, \ldots$.

(b) Show that $e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x$.

29. A mortality table is defined such that

$$ {}_tp_x = \left(1 - \frac{t}{100 - x}\right)^{0.5}$$

for $0 \leq x < 100$ and $0 \leq t < 100 - x$; and ${}_tp_x = 0$ for $t \geq 100 - x$.

(a) State the limiting age, ω.

(b) Calculate $\mathring{e}_{40}$.

(c) Calculate $\mathrm{Var}(T_{40})$.

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30. (a) Define 'selection effect'.

(b) You are given the following two quotations for a 10-year term life insurance:

Company	**X**	**Y**
Policyholder	Age 28, non-smoker	Age 28, non-smoker
Medical exam required?	Yes	No
Annual premium	\$120.00	\$138.00

(i) Explain the difference between the two premiums in laymen's terms.

(ii) Explain the difference between the two premiums in actuarial terms.

(c) You are given the following select-and-ultimate life table:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
65	0.01	0.04	0.07	67
66	0.03	0.06	0.09	68
67	0.05	0.08	0.12	69

(i) State the select period.

(ii) Calculate ${}_{1|2}q_{[65]+1}$.

(iii) Calculate ${}_{0.4}p_{[66]+0.3}$, assuming constant force of mortality between integer ages.

31. You are given the following life table:

x	l_x	x	l_x	x	l_x
91	27	94	12	97	3
92	21	95	8	98	1
93	16	96	5	99	0

(a) Calculate e_{91}.

(b) Calculate $\mathring{e}_{91}$, assuming uniform distribution of deaths between integer ages.

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32. You are given the following 4-year select-and-ultimate life table:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	$q_{[x]+3}$	q_{x+4}	$x+4$
40	0.00101	0.00175	0.00205	0.00233	0.00257	44
41	0.00113	0.00188	0.00220	0.00252	0.00293	45
42	0.00127	0.00204	0.00240	0.00280	0.00337	46
43	0.00142	0.00220	0.00262	0.00316	0.00384	47
44	0.00157	0.00240	0.00301	0.00367	0.00445	48

(a) Calculate the index of selection at age 44, $I(44, k)$ for $k = 0,1,2,3$.

(b) Construct the table of $l_{[x]+t}$, for $x = 40, 41, 42$ and for all t. Use $l_{[40]} = 10{,}000$.

(c) Calculate the following probabilities:

(i) ${}_2p_{[42]}$

(ii) ${}_3q_{[41]+1}$

(iii) ${}_{3|2}q_{[41]}$

33. You are given the following excerpt of a life table:

x	l_x
50	100,000
51	99,900
52	99,700
53	99,500
54	99,100
55	98,500

(a) Calculate d_{52}.

(b) Calculate ${}_{2|}q_{50}$.

(c) Assuming uniform distribution of deaths between integer ages, calculate the value of ${}_{4.3}p_{50.4}$.

(d) Assuming constant force of mortality between integer ages, calculate the value of ${}_{4.3}p_{50.4}$.

2.7 Solutions to Exercise 2

1. (a) ${}_2d_{52} = l_{52} - l_{54} = 99700 - 99100 = 600.$

   (b) ${}_{3|}q_{50} = \dfrac{d_{53}}{l_{50}} = \dfrac{l_{53} - l_{54}}{l_{50}} = \dfrac{99500 - 99100}{100000} = 0.004.$

2. Expressing ${}_{5|15}q_{10}$ in terms of l_x, we have

$$\begin{aligned} {}_{5|15}q_{10} &= \frac{l_{15} - l_{30}}{l_{10}} \\ &= \frac{10000e^{-0.05\times 15} - 10000e^{-0.05\times 30}}{10000e^{-0.05\times 10}} \\ &= 0.4109. \end{aligned}$$

3. (a) ${}_{0.2}p_{42} = 1 - {}_{0.2}q_{42} = 1 - 0.2q_{42} = 1 - 0.2 \times (1 - 9400/9700) = 0.993814.$

   (b) $$\begin{aligned} {}_{2.6}q_{41} &= 1 - {}_{2.6}p_{41} = 1 - {}_2p_{41} \times {}_{0.6}p_{43} = 1 - {}_2p_{41} \times (1 - 0.6q_{43}) \\ &= 1 - \frac{l_{43}}{l_{41}}\left(1 - 0.6 \times \frac{l_{43} - l_{44}}{l_{43}}\right) = 1 - \frac{9400}{9900}\left(1 - 0.6 \times \frac{9400 - 9000}{9400}\right) \\ &= 0.074747. \end{aligned}$$

   (c) Here, both subscripts are non-integers, so we need to use the trick. First, we compute ${}_{1.6}p_{40.9}$:

$$ {}_{0.9}p_{40} \times {}_{1.6}p_{40.9} = {}_{2.5}p_{40} $$

   Then, we have

$$\begin{aligned} {}_{1.6}p_{40.9} = \frac{{}_{2.5}p_{40}}{{}_{0.9}p_{40}} = \frac{{}_2p_{40}\ {}_{0.5}p_{42}}{{}_{0.9}p_{40}} = \frac{{}_2p_{40}\,(1 - 0.5q_{42})}{1 - 0.9q_{40}} &= \frac{\dfrac{9700}{10000}\left(1 - 0.5 \times \dfrac{300}{9700}\right)}{1 - 0.9 \times \dfrac{100}{10000}} \\ &= 0.963673. \end{aligned}$$

   Hence, ${}_{1.6}q_{40.9} = 1 - 0.963673 = 0.036327.$

4. (a) ${}_{0.2}p_{42} = (p_{42})^{0.2} = (9400/9700)^{0.2} = 0.993736.$

   (b) $$\begin{aligned} {}_{2.6}q_{41} &= 1 - {}_{2.6}p_{41} = 1 - {}_2p_{41} \times {}_{0.6}p_{43} = 1 - {}_2p_{41} \times (p_{43})^{0.6} \\ &= 1 - \frac{l_{43}}{l_{41}}\left(\frac{l_{44}}{l_{43}}\right)^{0.6} = 1 - \frac{9400}{9900}\left(\frac{9000}{9400}\right)^{0.6} = 0.074958. \end{aligned}$$

   (c) First, we consider ${}_{1.6}p_{40.9}$:

$$ {}_{1.6}p_{40.9} = {}_{0.1}p_{40.9} \times {}_{1.5}p_{41} = {}_{0.1}p_{40.9} \times p_{41} \times {}_{0.5}p_{42} = (p_{40})^{0.1} \times p_{41} \times (p_{42})^{0.5} $$

   Hence,

$$\begin{aligned} {}_{1.6}q_{40.9} = 1 - (p_{40})^{0.1}\,(p_{41})\,(p_{42})^{0.5} &= 1 - \left(\frac{l_{41}}{l_{40}}\right)^{0.1}\left(\frac{l_{42}}{l_{41}}\right)\left(\frac{l_{43}}{l_{42}}\right)^{0.5} \\ &= 1 - \left(\frac{9900}{10000}\right)^{0.1}\left(\frac{9700}{9900}\right)\left(\frac{9400}{9700}\right)^{0.5} = 0.036441 \end{aligned}$$

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5. By linear interpolation,

$$l_{40.3} = 0.7 \times l_{40} + 0.3 \times l_{41} = 0.7 \times 9{,}313{,}166 + 0.3 \times 9{,}287{,}264 = 9{,}305{,}395,$$

$$l_{41.7} = 0.3 \times l_{41} + 0.7 \times l_{42} = 0.3 \times 9{,}287{,}264 + 0.7 \times 9{,}259{,}571 = 9{,}267{,}879.$$

Therefore, ${}_{1.4}q_{40.3} = 1 - 9{,}267{,}879/9{,}305{,}395 = 0.004032$.

6. Expressing ${}_{15|20}q_{40}$ in terms of l_x, we have ${}_{15|20}q_{40} = \dfrac{l_{55} - l_{75}}{l_{40}}$.

From the table, we have $l_{40} = 60{,}500$. Since deaths are uniformly distributed over each 10-year span, we have

$$l_{55} = \frac{l_{50} + l_{60}}{2} = \frac{55{,}800 + 50{,}200}{2} = 53{,}000$$

$$l_{75} = \frac{l_{70} + l_{80}}{2} = \frac{44{,}000 + 36{,}700}{2} = 40{,}350$$

$${}_{15|20}q_{40} = \frac{53{,}000 - 40{,}350}{60{,}500} = 0.2091$$

7. ${}_{2|2}q_{[50]} = {}_2p_{[50]} - {}_4p_{[50]}$.

${}_2p_{[50]} = p_{[50]} \times p_{[50]+1} = 0.98 \times 0.96 = 0.9408$.

${}_4p_{[50]} = p_{[50]} \times p_{[50]+1} \times p_{52} \times p_{53} = 0.98 \times 0.96 \times 0.94 \times 0.93 = 0.8224$.

Hence, ${}_{2|2}q_{[50]} = 0.9408 - 0.8224 = 0.1184$.

8. (a) ${}_3p_{73} = l_{76}/l_{73} = 17{,}355/20{,}696 = 0.838568$.

(b) ${}_{4|}q_{[70]+1} = \dfrac{l_{[70]+5} - l_{[70]+6}}{l_{[70]+1}} = \dfrac{l_{75} - l_{76}}{l_{[70]+1}} = \frac{18{,}508-17{,}355}{22{,}200} = 0.05194$.

(c) By linear interpolation,

$$l_{[70]+0.7} = 0.3 \times l_{[70]} + 0.7 \times l_{[70]+1} = 0.3 \times 22{,}507 + 0.7 \times 22{,}200 = 22{,}292.1,$$

$$l_{[70]+1.2} = 0.8 \times l_{[70]+1} + 0.2 \times l_{[70]+2} = 0.8 \times 22{,}200 + 0.2 \times 21{,}722 = 22{,}104.4.$$

Hence, ${}_{0.5}p_{[70]+0.7} = 22{,}104.4/22{,}292.1 = 0.991580$.

(d)
$$\begin{aligned} {}_{0.5}p_{[70]+0.7} &= {}_{0.3}p_{[70]+0.7} \times {}_{0.2}p_{[70]+1} = (p_{[70]})^{0.3}(p_{[70]+1})^{0.2} \\ &= \left(\frac{l_{[70]+1}}{l_{[70]}}\right)^{0.3}\left(\frac{l_{72}}{l_{[70]+1}}\right)^{0.2} = 0.991562. \end{aligned}$$

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9. We begin with finding $S_0(t)$ for $0 \le t \le 20$:

$$S_0(t) = \int_t^{20} f_0(u)\mathrm{d}u = \frac{\int_t^{20}(20-u)\mathrm{d}u}{200} = -\frac{\left[(20-u)^2\right]_t^{20}}{400} = \frac{(20-t)^2}{400}.$$

$$_tp_5 = \frac{S_0(t+5)}{S_0(5)} = \frac{(15-t)^2}{15^2} = \left(1-\frac{t}{15}\right)^2, \text{ for } 0 \le t \le 15.$$

$$\mathring{e}_5 = \int_0^{15} {}_tp_5 \ \mathrm{d}t = \int_0^{15}\left(1-\frac{t}{15}\right)^2 \ \mathrm{d}t = -\frac{15}{3}\left[\left(1-\frac{t}{15}\right)^3\right]_0^{15} = 5.$$

10. Since the survival function changes at $t = 30$, we need to decompose the integral into two parts.

$$\begin{aligned} \mathrm{E}\left(T_0\right) &= \int_0^\infty S_0(t)\mathrm{d}t \\ &= \int_0^{30}\left(1-\frac{t}{100}\right)\mathrm{d}t + \int_{30}^\infty 0.7e^{-0.02(t-30)}\mathrm{d}t \\ &= \left[t-\frac{t^2}{200}\right]_0^{30} + 0.7\int_0^\infty e^{-0.02u}\mathrm{d}u \\ &= 25.5 + \frac{0.7}{0.02} \\ &= 60.5 \end{aligned}$$

11. We begin with the calculation of $_tp_0$:

$$_tp_0 = S_0(t) = e^{-\int_0^t \mu_u \mathrm{d}u} = e^{-\int_0^t \frac{2u}{400-u^2}\mathrm{d}u} = e^{\ln\left(400-u^2\right)|_0^t} = e^{\ln\left(\frac{400-t^2}{400}\right)} = 1-\frac{t^2}{400}$$

$$\mathrm{E}\left(T_0\right) = \int_0^{20} {}_tp_0 \ \mathrm{d}t = \int_0^{20}\left(1-\frac{t^2}{400}\right)\mathrm{d}t = \left(t-\frac{t^3}{1200}\right)\Big|_0^{20} = \frac{40}{3}$$

$$\mathrm{E}(T_0^2) = 2\int_0^{20} t \ {}_tp_0 \ \mathrm{d}t = 2\int_0^{20}\left(t-\frac{t^3}{400}\right)\mathrm{d}t = 2\left(\frac{t^2}{2}-\frac{t^4}{1600}\right)\Big|_0^{20} = 200$$

$$\mathrm{Var}(T_0) = \mathrm{E}(T_0^2) - \left[\mathrm{E}(T_0)\right]^2 = 200 - \left(\frac{40}{3}\right)^2 = 22.22.$$

12. Here, the lifetime follows De Moivre's law (i.e., a uniform distribution). By using the properties of uniform distributions, we immediately obtain

$$\begin{aligned} \mathrm{Var}(T_0) = 468.75 &= \frac{\omega^2}{12} \\ \omega^2 &= 5625 \\ \omega &= 75 \end{aligned}$$

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13. When $\mu_x = \mu$ for all $x \geq 0$, the lifetime follows an exponential distribution. Using the properties of exponential distributions, we immediately obtain

$$\mathring{e}_{30} = 40 = \frac{1}{\mu} \quad \Rightarrow \quad \mu = \frac{1}{40} = 0.025.$$

Also, we know that when $\mu_x = \mu$ for all $x \geq 0, {}_tp_x = e^{-\mu t}$. Hence,

$${}_5p_{20} = e^{-0.025\times 5} = 0.8825.$$

14. First, we calculate $S_0(t)$

$$S_0(t) = \frac{l_t}{l_0} = \frac{10000 - t^2}{10000} = 1 - \frac{t^2}{10000}.$$

Then,

$$\mathrm{E}(T_0) = \int_0^{100} S_0(t)\mathrm{d}t = \int_0^{100} \left(1 - \frac{t^2}{10000}\right) \mathrm{d}t = \left(t - \frac{t^3}{30000}\right)\Bigg|_0^{100} = 66.6667,$$

and

$$\mathrm{E}(T_0^2) = 2\int_0^{100} tS_0(t)\mathrm{d}t = 2\int_0^{100} \left(t - \frac{t^3}{10000}\right) \mathrm{d}t = 2\left(\frac{t^2}{2} - \frac{t^4}{40000}\right)\Bigg|_0^{100} = 5000.$$

Hence, $\mathrm{Var}\,(T_0) = \mathrm{E}\left(T_0^2\right) - [\mathrm{E}\,(T_0)]^2 = 5000 - 66.6667^2 = 555.6$.

15. Since $\mu_x = 0.02$ for all $x \geq 0$, we immediately have ${}_tp_x = e^{-0.02t}$. Then,

$$\mathring{e}_{10:\overline{10|}} = \int_0^{10} {}_tp_{10}\ \mathrm{d}t = \int_0^{10} e^{-0.02t}\ \mathrm{d}t = -\left[\frac{1}{0.02}e^{-0.02t}\right]_0^{10} = 9.063.$$

16. First, we obtain ${}_tp_{20}$ as follows:

$${}_tp_{20} = \frac{S_0(20+t)}{S_0(20)} = 1 - \frac{t}{\omega - 20}.$$

Then, we have

$$\mathring{e}_{20:\overline{30|}} = \int_0^{30} {}_tp_{20}\ \mathrm{d}t = \int_0^{30} \left(1 - \frac{t}{\omega-20}\right) \mathrm{d}t = \left[t - \frac{t^2}{2(\omega-20)}\right]_0^{30} = 30 - \frac{450}{\omega-20} = 22.5.$$

This gives $\omega = 80$.

Note that the underlying lifetime follows De Moivre's law. This implies that T_{30} is uniformly distributed over the interval $[0, \omega - 30)$, that is, [0,50). Using the properties of uniform distributions, we have $\mathrm{Var}\,(T_{30}) = 50^2/12 = 208.33$.

17. First, we compute ${}_tp_5$:

$${}_tp_5 = \frac{l_{5+t}}{l_5} = \frac{80 - (5+t)}{80-5} = 1 - \frac{t}{75}.$$

Hence,

$$\mathring{e}_{5:\overline{15|}} = \int_0^{15} \left(1 - \frac{t}{75}\right) \mathrm{d}t = \left[t - \frac{t^2}{150}\right]_0^{15} = 13.5.$$

18. Since mortality follows De Moivre's law, for 20 year olds, future lifetime follows a uniform distribution over $[0, \omega - 20)$. We have

$$\mathring{e}_{20} = 30 = \frac{\omega - 20}{2},$$

which gives $\omega = 80$. Since death occurs uniformly over [0,60), we have $q_{20} = 1/60$. Hence, the answer is (A).

19. The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 80.

- Suppose that there are l_{30} persons in the entire population initially. At time 0 (i.e., at age 30), there are $0.5l_{30}$ nonsmokers and $0.5l_{30}$ smokers.
- For nonsmokers, the proportion of individuals who can survive to age 80 is $e^{-0.08\times 50} = e^{-4}$. For smokers, the proportion of individuals who can survive to age 80 is $e^{-0.16\times 50} = e^{-8}$. As a result, at age 80, there are $0.5l_{30}e^{-4}$ nonsmokers and $0.5l_{30}e^{-0.8}$ smokers. Hence, among those who can survive to age 80,

$$\frac{0.5l_{30}e^{-4}}{0.5l_{30}e^{-4} + 0.5l_{30}e^{-8}} = \frac{1}{1 + e^{-4}} = 0.982014$$

are nonsmokers and $1 - 0.982014 = 0.017986$ are smokers.

Second, we need to calculate q_{80} for both smokers and nonsmokers.

- For a nonsmoker at age 80, $q_{80}^{n} = 1 - e^{-0.08}$.
- For a smoker at age 80, $q_{80}^{s} = 1 - e^{-0.16}$.

Finally, for the whole population, we have

$$q_{80} = 0.982014(1 - e^{-0.08}) + 0.017986(1 - e^{-0.16}) = 0.07816.$$

Hence, the answer is (A).

20. The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 60.

- Suppose that there are l_0 persons in the entire population initially. At time 0 (i.e., at age 0), there are $0.5l_0$ males and $0.5l_0$ females.
- For males, the proportion of individuals who can survive to age 60 is $e^{-0.10\times 60} = e^{-6}$. For females, the proportion of individuals who can survive to age 60 is $e^{-0.08\times 60} = e^{-4.8}$. As a result, at age 60, there are $0.5l_0e^{-6}$ males and $0.5l_0e^{-4.8}$ females. Hence, among those who can survive to age 60,

$$\frac{0.5l_0e^{-6}}{0.5l_0e^{-6} + 0.5l_0e^{-4.8}} = \frac{1}{1 + e^{1.2}} = 0.231475$$

are males and $1 - 0.231475 = 0.768525$ are females.

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Second, we need to calculate q_{60} for both males and females.

- For a male at age 60, $q_{60}^m = 1 - e^{-0.10}$.
- For a female at age 60, $q_{60}^f = 1 - e^{-0.08}$.

Finally, for the whole population, we have

$$q_{60} = 0.231475(1 - e^{-0.10}) + 0.768525(1 - e^{-0.08}) = 0.0811.$$

Hence, the answer is (B).

21. From Statement (ii), we know that the underlying lifetime follows De Moivre's law. By using the properties of uniform distributions, we immediately have

$$\mathring{e}_0 = \frac{\omega}{2} = 25,$$

which gives $\omega = 50$.

Under De Moivre's law, T_{10} is uniformly distributed over the interval $[0, \omega - 10)$, that is, [0,40). By using the properties of uniform distributions, we immediately obtain

$$\text{Var}\,(T_{10}) = 40^2/12 = 133.3.$$

Hence, the answer is (C).

22. From the given survival function, we know that the underlying lifetime follows De Moivre's law. First, we find ${}_tp_{30}$:

$${}_tp_{30} = \frac{S_0(30+t)}{S_0(30)} = 1 - \frac{t}{\omega - 30}.$$

We then use Statement (i) to find ω:

$$\begin{aligned}\mathring{e}_{30:\overline{40|}} &= \int_0^{40} {}_tp_{30}\ \text{d}t = \int_0^{40} \left(1 - \frac{t}{\omega - 30}\right) \text{d}t = \left[t - \frac{t^2}{2(\omega - 30)}\right]_0^{40} \\ &= 40 - \frac{40^2}{2(\omega - 30)} = 27.692.\end{aligned}$$

This gives $\omega = 95$.

Under De Moivre's law, T_{30} is uniformly distributed over $[0, \omega - 30)$, that is [0,65). By using the properties of uniform distributions, we immediately obtain

$$\text{Var}(T_{30}) = 65^2/12 = 352.1.$$

Hence the answer is (B).

23. For the original model, $S_0(t) = (1 - t/\omega)^\alpha$. This gives

$$\text{E}(T_0) = \int_0^\omega S_0(t)\text{d}t = \int_0^\omega \left(1 - \frac{t}{\omega}\right)^\alpha \text{d}t = -\frac{\omega}{\alpha+1}\left(1 - \frac{t}{\omega}\right)^{\alpha+1}\Bigg|_0^\omega = \frac{\omega}{\alpha+1},$$

and

$$\mu_x = -\frac{S_0'(x)}{S_0(x)} = \frac{\frac{\alpha}{\omega}\left(1 - \frac{x}{\omega}\right)^{\alpha-1}}{\left(1 - \frac{x}{\omega}\right)^\alpha} = \frac{\alpha}{\omega - x}.$$

Let α and α^* be the original and new values of α, respectively. Since the new complete expectation of life is half what it was previously, we have

$$\frac{\omega}{\alpha^*+1} = \frac{1}{2}\left(\frac{\omega}{\alpha+1}\right), \text{ or } 2(\alpha+1) = \alpha^*+1.$$

Also, since the new force of mortality is 2.25 times the previous force of mortality for all durations, we have

$$\frac{\alpha^*}{\omega - x} = \frac{2.25\alpha}{\omega - x},$$

or $\alpha^* = 2.25\alpha$. Solving $2(\alpha+1) = 2.25\alpha + 1$, we obtain $\alpha = 4$. Hence, the answer is (D).

24. The primary objective of this question is to examine your knowledge on the recursion formula $e_x = p_x(1 + e_{x+1})$.

Note that M and N have the same force of mortality from age 26. This means that

$${}_kp_{26}^M = {}_kp_{26}^N, \quad k = 1,2,3,\ldots,$$

and consequently that

$$e_{26}^M = e_{26}^N.$$

Using the identity above, we have

$$e_{25}^N = p_{25}^N(1 + e_{26}^N) = p_{25}^N(1 + e_{26}^M).$$

We can find p_{25}^N using the force of mortality given:

$$\begin{aligned} p_{25}^N &= \exp\left(-\int_0^1 \mu_{25+t}^N \mathrm{d}t\right) \\ &= \exp\left(-\int_0^1 (\mu_{25+t}^M + 0.1(1-t))\mathrm{d}t\right) \\ &= e^{-\int_0^1 \mu_{25+t}^M \mathrm{d}t} e^{-\int_0^1 0.1(1-t)\mathrm{d}t} \\ &= p_{25}^M \exp\left([0.05(1-t)^2]_0^1\right) \\ &= p_{25}^M e^{-0.05}. \end{aligned}$$

This implies that

$$e_{25}^N = e^{-0.05} p_{25}^M (1 + e_{26}^M) = e^{-0.05} e_{25}^M = 0.951 \times 10 = 9.51.$$

Hence, the answer is (D).

25. Splicing two functions $h(x)$ and $g(x)$ on an interval $[a,b]$ means that we break up the interval into two smaller intervals $[a,c]$ and $(c,d]$ and define the spliced function to equal $h(x)$ on $[a,c]$ and $g(x)$ on $(c,d]$. In this case, we are breaking up [0,100] into [0,50] and (50,100]. Our new function will equal $kf^a(t)$ on [0,50] and $1.2f^b(t)$ on (50, 100]. The spliced function needs to be a density function on [0, 100], so we need to find the value of k that makes the total area under the curve equal 1.

We will start by looking at $1.2f^b(t)$. For a De Moivre's model with $\omega = 100$, $f^b(t) = 1/100$ which means $1.2f^b(t) = 1.2/100$. Thus, the area under the curve (50,100] is

$$\int_{50}^{100} \frac{1.2}{100}\,\mathrm{d}t = \frac{1.2(100-50)}{100} = 0.6.$$

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This means that the area under the curve on [0, 50] must be 0.4. So,

$$k\int_0^{50} f^a(t)\mathrm{d}t = k\left(\frac{l_0 - l_{50}}{l_0}\right) = k\frac{1049099}{10000000} = 0.4,$$

where l_t is the life function that corresponds to the Illustrative Life Table. This gives $k = 3.8128$.

Let ${}_tq_0{}^*$ be death probabilities that corresponds to the Illustrative Life Table. Then

$$\begin{aligned}
{}_{10}p_{40} =& \frac{{}_{50}p_0}{{}_{40}p_0} = \frac{1 - {}_{50}q_0}{1 - {}_{40}q_0}\\
=& \frac{1 - \int_0^{50} kf^a(t)\mathrm{d}t}{1 - \int_0^{40} kf^a(t)\mathrm{d}t} = \frac{1 - k\ {}_{50}q_0{}^*}{1 - k\ {}_{40}q_0{}^*}\\
=& \frac{1 - 0.4}{1 - k\frac{686834}{10000000}} = \frac{0.6}{0.738124} = 0.8129.
\end{aligned}$$

Hence, the answer is (A).

26. (a) The proof is as follows:

$$\begin{aligned}
e_x &= \sum_{k=1}^{\infty} {}_kp_x = p_x + \sum_{k=2}^{\infty} {}_kp_x\\
&= p_x + p_x \sum_{k=2}^{\infty} {}_{k-1}p_{x+1} = p_x + p_x \sum_{j=1}^{\infty} p_{x+1}\\
&= p_x + p_x e_{x+1} = p_x\left(1 + e_{x+1}\right)
\end{aligned}$$

(b) Under UDD, $T_x = K_x + U$, where U follows a uniform distribution over the interval [0, 1]. Taking expectation on both sides, we have $\mathrm{E}(T_x) = \mathrm{E}(K_x) + \mathrm{E}(U)$, which implies

$$\mathring{e}_x = e_x + \frac{1}{2}.$$

(c) This is a difficult question. To answer this question, you need to use the following three facts:

- For a one-year select period, $l_{[x]+1} = l_{x+1} = l_{[x]} - d_{[x]}$ and $e_{[x]+1} = e_{x+1}$.
- $e_x = p_x(1 + e_{x+1})$
- Under UDD, $\mathring{e}_x = e_x + \frac{1}{2}$

We can complete the second last column of the table by using $l_{[x]+1} = l_{x+1} = l_{[x]} - d_{[x]}$:

$$l_{81} = l_{[80]} - d_{[80]} = 1000 - 90 = 910,$$

$$l_{82} = l_{[81]} - d_{[81]} = 920 - 90 = 830.$$

Under UDD, we have $e_{[80]} = 8.5 - 0.5 = 8$.

We then apply the recursion formula as follows:

$$e_{[80]} = p_{[80]}(1 + e_{[80]+1}) = p_{[80]}\left(1 + e_{81}\right) = \frac{910}{1000}\left(1 + e_{81}\right).$$

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This gives $e_{81} = 7.791208791$.

Also, we have

$$e_{[81]} = p_{[81]}(1 + e_{82}), \quad e_{81} = p_{81}(1 + e_{82}).$$

This means

$$\begin{aligned} e_{[81]} &= e_{81}\frac{p_{[81]}}{p_{81}} = e_{81}\frac{l_{[81]+1}/l_{[81]}}{l_{82}/l_{81}} = e_{81}\frac{l_{82}/l_{[81]}}{l_{82}/l_{81}} \\ &= e_{81}\frac{l_{81}}{l_{[81]}} = 7.791208791 \times \frac{910}{920} = 7.7065. \end{aligned}$$

Finally, assuming UDD, $\mathring{e}_{[81]} = 7.7065 + 0.5 = 8.2065$.

27. (a) $S_x(t) = \dfrac{S_0(x+t)}{S_0(x)} = \dfrac{e^{-0.02(x+t)}}{e^{-0.02x}} = e^{-0.02t}$

(b) $\mu_x = \dfrac{-S_0'(x)}{S_0(x)} = \dfrac{0.02e^{-0.02x}}{e^{-0.02x}} = 0.02$

(c) $\mathring{e}_{10:\overline{10}|} = \displaystyle\int_0^{10} {}_tp_{10}\mathrm{d}t = \int_0^{10} e^{-0.02t}\mathrm{d}t = \left.\frac{e^{-0.02t}}{-0.02}\right|_0^{10} = 9.06346$

(d) Since ${}_kp_x = S_x(k) = e^{-0.02k}$, we have

$$e_x = \sum_{k=1}^{\infty} {}_kp_x = \sum_{k=1}^{\infty} e^{-0.02k} = \frac{e^{-0.02}}{1 - e^{-0.02}} = 49.5017.$$

28. (a) The event $K_x = k$ is the same as $k \le T_x < k+1$, which means the individual cannot die within the first k years and must die during the subsequent year. The probability associated with this event must be ${}_{k|}q_x$, the k-year deferred one-year death probability.

(b) The proof is as follows:

$$\begin{aligned} e_{x:\overline{n}|} &= \sum_{k=0}^{n-1} k\Pr(K_x = k) + \sum_{k=n}^{\infty} n\Pr(K_x = k) \\ &= \sum_{k=0}^{n-1} k \times {}_{k|}q_x + \sum_{k=n}^{\infty} n \times {}_{k|}q_x \\ &= 0 + 1 \times {}_{1|}q_x + 2 \times {}_{2|}q_x + \ldots + (n-1) \times {}_{n-1|}q_x \\ &\quad + n \times {}_{n|}q_x + n \times {}_{n+1}q_x + \ldots \\ &= (p_x - {}_2p_x) + 2({}_2p_x - {}_3p_x) + \ldots + (n-1)({}_{n-1}p_x - {}_np_x) + \ldots \\ &\quad + n({}_np_x - {}_{n+1}p_x) + n({}_{n+1}p_x - {}_{n+2}p_x) + n({}_{n+2}p_x - {}_{n+3}p_x) + \ldots \\ &= p_x + {}_2p_x + {}_3p_x + \ldots + {}_np_x \\ &= \sum_{k=1}^{n} {}_kp_x \end{aligned}$$

29. (a) $\omega = 100$

(b) $\mathring{e}_{40} = \int_0^{100-40} {}_tp_{40}\, dt = \int_0^{60} \left(1 - \frac{t}{60}\right)^{0.5} dt$

$$= \frac{-60}{1.5}\left(1 - \frac{t}{60}\right)^{1.5}\Bigg|_0^{60} = 0 + \frac{60}{1.5} = 40$$

(c) $E(T_{40}^2) = 2\int_0^{60} t\left(1 - \frac{t}{60}\right)^{0.5} dt.$

Let $y = 1 - t/60$. We have

$$\begin{aligned} E(T_{40}^2) &= 2\int_0^{60} 60(1-y)y^{0.5}(-60)dy \\ &= 7200\int_0^1 \left(y^{1/2} - y^{3/2}\right) dy \\ &= 7200\left[\frac{2}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1 = 1920 \end{aligned}$$

Hence, $\text{Var}(T_{40}) = 1920 - 40^2 = 320.$

30. (a) The effect of medical (or other) evidence at the inception of an insurance contract.

(b) (i) In laymen's terms:

- Company Y requires no medical examination, so it is taking more risk.
- Company Y has a higher change of adverse selection.
- Company Y has to charge more premium to compensate for the additional risk.

(ii) In actuarial terms:

- Company X requires a medical examination, which means there is a stronger effect of selection.
- The index of selection is higher (closer to 1).
- The death probabilities used to price the policy are lower. This means the premium charged by Company X is lower than that charged by Company Y.

(c) (i) 2 years.

(ii)
$$\begin{aligned} {}_{1|2}q_{[65]+1} &= p_{[65]+1} \times {}_2q_{[65]+2} \\ &= p_{[65]+1} \times {}_2q_{67} \\ &= p_{[65]+1}\left[1 - (1-q_{67})(1-q_{68})\right] \\ &= (1-0.04)[1-(1-0.07)(1-0.09)] = 0.147552. \end{aligned}$$

(iii) ${}_{0.4}p_{[66]+0.3} = (p_{[66]})^{0.4} = (1-0.03)^{0.4} = 0.987890.$

31. (a) $e_{91} = p_{91} + {}_2p_{91} + {}_3p_{91} + \ldots = (l_{92} + l_{93} + \ldots + l_{99})/l_{91} = 2.44$ years.

(b) Under UDD $\mathring{e}_{91} = e_{91} + 0.5 = 2.94.$

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32. (a) $I(44{,}0) = 1 - \dfrac{q_{[44]}}{q_{44}} = 1 - \dfrac{0.00157}{0.00257} = 0.38911$

$I(44{,}1) = 1 - \dfrac{q_{[44]+1}}{q_{44+1}} = 1 - \dfrac{0.00240}{0.00293} = 0.18089$

$I(44{,}2) = 1 - \dfrac{q_{[44]+2}}{q_{44+2}} = 1 - \dfrac{0.00301}{0.00337} = 0.10682$

$I(44{,}3) = 1 - \dfrac{q_{[44]+3}}{q_{44+3}} = 1 - \dfrac{0.00367}{0.00384} = 0.04427$

Comment: The index of selection reduces as the value of k increases. This agrees with the fact that as duration increases, selection effect tapers off.

(b) The table is calculated as follows:

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}	$x+4$
40	10000	9989.9	9972.4	9952.0	9928.8	44
41	9980.2	9969.0	9950.2	9928.3	9903.3	45
42	9958.8	9946.1	9925.8	9902.0	9987.3	46

(c) (i) ${}_2p_{[42]} = l_{[42]+2}/l_{[42]} = 0.99669$

(ii) ${}_3q_{[41]+1} = (l_{[41]+1} - l_{[41]+4})/l_{[41]+1} = 0.00659$

(iii) ${}_{3|2}q_{[41]} = (l_{[41]+3} - l_{[41]+5)})/l_{[41]} = 0.00542.$

33. (a) $d_{52} = l_{52} - l_{53} = 99700 - 99500 = 200$

(b) ${}_{2|}q_{50} = {}_2p_{50}q_{52} = \dfrac{l_{52}}{l_{50}}\dfrac{d_{52}}{l_{52}} = \dfrac{d_{52}}{l_{50}} = \dfrac{200}{100000} = 0.002$

(c) Since ${}_{0.4}p_{50}\ {}_{4.3}p_{50.4} = {}_{4.7}p_{50}$, we have

$$ {}_{4.3}p_{50.4} = \frac{{}_{4.7}p_{50}}{{}_{0.4}p_{50}} = \frac{{}_4p_{50} \times {}_{0.7}p_{54}}{{}_{0.4}p_{50}} = \frac{{}_4p_{50}\,(1 - 0.7q_{54})}{1 - 0.4q_{50}} = \frac{\frac{l_{54}}{l_{50}}\left(1 - 0.7\frac{d_{54}}{l_{54}}\right)}{1 - 0.4\frac{d_{50}}{l_{50}}}. $$

Substituting, we obtain ${}_{4.3}p_{50.4} = 0.987195.$

(d)
$$ \begin{aligned} {}_{4.3}p_{50.4} &= {}_{0.6}p_{50} \times {}_{3.7}p_{51} = {}_{0.6}p_{50.4} \times {}_3p_{51} \times {}_{0.7}p_{54} \\ &= (p_{50})^{0.6}\ {}_3p_{51}\ (p_{54})^{0.7} \\ &= \left(\frac{99900}{100000}\right)^{0.6}\left(\frac{99100}{99900}\right)\left(\frac{98500}{99100}\right)^{0.7} = 0.987191 \end{aligned} $$

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Mock Tests

There are 34 multiple choice questions on Exam FAM and the time limit is 3.5 hours. In each of the five mock tests, there are 17 questions and you should aim at spending 1 hr 45 minutes on it.

Mock Test 1

****BEGINNING OF EXAMINATION****

1. For a whole life insurance of 1,000 issued to life selected at age x,

   (i) Percent of premium expenses is 90% in the first year, and 10% in each year thereafter.

   (ii) Maintenance expenses are 15 per 1,000 of insurance in the first year, and 3 per 1,000 of insurance thereafter.

   (iii) Claim settlement expenses are 10 per 1,000 of insurance.

   (iv) A 15-year select and ultimate mortality is to be used.

   Determine the expression of the gross premium for the policy using equivalence principle.

   (A) $\dfrac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.9}$

   (B) $\dfrac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}$

   (C) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{\ddot{a}_{[x]} - 0.9}$

   (D) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.9}$

   (E) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}$

2. You are given:

   (i) $i = 0.07$

   (ii) $\ddot{a}_x = 11.7089$

   (iii) $\ddot{a}_{x:\overline{40}|} = 11.55$

   (iv) $\ddot{a}_{x+40} = 7.1889$

   Find $A^{1}_{x:\overline{40}|}$.

   (A) 0.120 (B) 0.147 (C) 0.172 (D) 0.197 (E) 0.222

3. You are given the following select-and-ultimate table:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
65	0.11	0.13	0.15	67
66	0.12	0.135	0.16	68
67	0.13	0.145	0.17	69

   Deaths are uniformly distributed over each year of age.

   Find ${}_{1.8}p_{[66]+0.6}$.

   (A) 0.69 (B) 0.71 (C) 0.73 (D) 0.75 (E) 0.77

4. For a 3-year fully discrete endowment insurance of 1,000 on (x), you are given:

   (i) $q_x = 0.1$

   (ii) $q_{x+1} = 0.15$

   (iii) $v = 0.9$

   (iv) Deaths are uniformly distributed over each year of age.

   Calculate the net premium policy value 9 months after the issuance of the policy.

   (A) 274 (B) 280 (C) 283 (D) 286 (E) 290

5. Which of the following statements is/are correct?

   I. Insurable interest in an entity exists if one would suffer a financial loss if that entity is damaged.

   II. Insurable interest is related to the concept of adverse selection.

   III. Stranger owned life insurance is illegal in many jurisdiction because the purchaser has no insurable interest in the insured.

   (A) I only (B) II only (C) I and III only
   (D) II and III only (E) I, II and III

6. Which of the following is/are strictly increasing function(s) of T_x for all $T_x \geq 0$?

   I. The present value random variable for a continuous whole life annuity of \$1 on (x)

   II. The present value random variable for a continuous n-year temporary life annuity of \$1 on (x)

   III. The net future loss at issue random variable for a fully continuous whole life insurance of \$1 on (x)

   (A) I only (B) III only (C) I, II only
   (D) I, III only (E) I, II and III

7. For a fully discrete whole life insurance on (30), you are given:

   (i) $i = 0.05$

   (ii) $q_{29+h} = 0.004$

   (iii) The net amount at risk for policy year h is 1295.

   (iv) The terminal policy value for policy year $h-1$ is 179.

   (v) $\ddot{a}_{30} = 16.2$

   Calculate the initial policy value for policy year $h+1$.

   (A) 188 (B) 192 (C) 200 (D) 214 (E) 226

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8. You are given:

(i) $\mu_{x+t} = \begin{cases} 0.02 & 0 \le t < 1 \\ 0.07 & 1 \le t < 2 \end{cases}$ (ii) $Y = \min(T_x, 2)$

Calculate $\mathrm{E}(Y)$.

(A) 1.88 (B) 1.90 (C) 1.92 (D) 1.94 (E) 1.96

9. An insurer issues fully discrete whole life insurance policies to a group of 50 high-risk drivers all aged 35 with a sum insured of 2,000. You are given:

(i) The mortality of each driver follows the Standard Ultimate Life Table with an age rating of 3 years. That is,

$$q_x = q_{x+3}^{SULT},$$

where q_y^{SULT} is the 1-year death probability under the Standard Ultimate Life Table.

(ii) Lifetimes of the group of 50 high-risk drivers are independent.

(iii) Premiums are payable annually in advance. Each premium is 110% of the net annual premium.

(iv) $i = 0.05$

By assuming a normal approximation, estimate the 95^{th}-percentile of the net future loss random variable.

(A) 1090 (B) 1240 (C) 1390 (D) 1540 (E) 1690

10. Which of the following regarding is/are correct?

I. Variable annuity has cash value.

II. Term insurance has cash value.

III. Universal life insurance has cash value.

(A) I only (B) II only (C) III only

(D) II and III only (E) I, II and III

11. You are given:

(i) $\delta = 0.05$

(ii) $\bar{A}_x = 0.44$

(iii) ${}^2\bar{A}_x = 0.22$

Consider a portfolio of 100 fully continuous whole life insurances. The ages of the all insureds are x, and their lifetimes are independent. The face amount of the policies, the premium rate and the number of policies are as follows:

Face amount	Premium rate	Number of Policies
100	4.3	75
400	17.5	25

By using a normal approximation, calculate the probability that the present value of the aggregate loss-at-issue for the insurer's portfolio will exceed 700.

(A) $1 - \Phi(2.28)$ (B) $1 - \Phi(0.17)$ (C) $\Phi(0)$
(D) $\Phi(0.17)$ (E) $\Phi(2.28)$

12. Victor is now age 22, and his future lifetime has the following cumulative distribution function:

$$\mathrm{F}_{22}(t) = 1 - (1 + 0.04t)e^{-0.04t}.$$

Let Z be the present value random variable for a fully continuous life insurance that pays 100 immediately on the death of Victor provided that he dies between ages 32 and 52.

The force of interest is 0.06.

Find the 70^{th}-percentile of Z.

(A) 0 (B) 25 (C) 40 (D) 47 (E) 50

13. You are given:

(i) $i = 0.05$

(ii) $A_{60} = 0.560$

(iii) $A_{40} = 0.176$

(iv) ${}_{20}p_{40} = 0.75$

Calculate $\ddot{a}^{(3)}_{40:\overline{20|}}$ using the two-term Woolhouse's approximation.

(A) 13.7 (B) 14.2 (C) 14.5 (D) 14.9 (E) 15.1

14. You are given:

(i) $\delta = 0.05$

(ii) $\ddot{a}_{60} = 12.18$

(iii) $p_{60} = 0.98$

Using the claims accelerated approach, calculate $\ddot{a}_{61}^{(6)}$.

(A) 11.6 (B) 11.7 (C) 11.8 (D) 11.9 (E) 12.1

15. Let Y be the present value random variable for a special three-year temporary life annuity on (x). You are given:

(i) The life annuity pays $2 + k$ at time k, for $k = 0$, 1 and 2.

(ii) $v = 0.9$

(iii) $p_x = 0.8$, $p_{x+1} = 0.75$, $p_{x+2} = 0.5$

Calculate the standard deviation of Y.

(A) 1.2 (B) 1.8 (C) 2.4 (D) 3.0 (E) 3.6

16. You are given:

$$\mu_x = \begin{cases} 0.04 & 50 \le x < 60 \\ 0.05 + 0.001(x-60)^2 & 60 \le x < 70 \end{cases}$$

Calculate ${}_{4|14}q_{50}$.

(A) 0.38 (B) 0.44 (C) 0.47 (D) 0.50 (E) 0.56

17. You are given:

(i) $p_{41} = 0.999422$

(ii) $i = 0.03$

(iii) $\ddot{a}_{42:\overline{23}|} = 16.7147$

Calculate the Full Preliminary Term reserve at time 2 for a 25-year fully discrete endowment insurance, issued to (40), with sum insured 75,000.

(A) 2,188 (B) 2,190 (C) 2,192 (D) 2,194 (E) 2,196

**** END OF EXAMINATION ****

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Solutions to Mock Test 1

Question #	Answer
1	B
2	E
3	E
4	D
5	C
6	A
7	E
8	D
9	B
10	C

Question #	Answer
11	A
12	A
13	B
14	A
15	C
16	C
17	D

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1. [Chapter 7] Answer: (B)

Let G be the gross premium. By the equivalence principle,

APV of gross premiums = APV of death benefit + APV of expenses.

$$\begin{aligned}G\ddot{a}_{[x]} &= 1010\bar{A}_{[x]} + 0.9G + 0.1Ga_{[x]} + 15 + 3a_{[x]}\\ &= 1010\bar{A}_{[x]} + 0.9G + 0.1G(\ddot{a}_{[x]} - 1) + 15 + 3(\ddot{a}_{[x]} - 1)\\ G &= \frac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}\end{aligned}$$

2. [Chapter 4] Answer: (E)

We first change all annuities into insurances.

Statement (ii) implies that $A_x = 1 - \frac{0.07}{1.07} \times 11.7089 = 0.2340.$

Statement (iii) implies that $A_{x:\overline{40}|} = 1 - \frac{0.07}{1.07} \times 11.55 = 0.2444.$

Statement (iv) implies that $A_{x+40} = 1 - \frac{0.07}{1.07} \times 7.1889 = 0.5297.$

Finally, by $A_x = A^1_{x:\overline{40}|} + A_{x:\overset{\;1}{\overline{40}|}}A_{x+40}$ and $A_{x:\overset{\;1}{\overline{40}|}} + A^1_{x:\overline{40}|} = A_{x:\overline{40}|}$,

$$0.2340 = A^1_{x:\overline{40}|} + (0.2444 - A^1_{x:\overline{40}|}) \times 0.5297$$

On solving, we get $A^1_{x:\overline{40}|} = \dfrac{0.2340 - 0.2444 \times 0.5297}{1 - 0.5297} = 0.2223.$

3. [Chapter 2] Answer: (E)

Method 1: $$\begin{aligned}{}_{1.8}p_{[66]+0.6} &= {}_{0.4}p_{[66]+0.6} \times {}_{1.4}p_{[66]+1}\\ &= {}_{0.4}p_{[66]+0.6} \times p_{[66]+1} \times {}_{0.4}p_{[66]+2}\\ &= {}_{0.4}p_{[66]+0.6} \times p_{[66]+1} \times {}_{0.4}p_{68}\end{aligned}$$

Obviously, $p_{[66]+1} = 1 - 0.135 = 0.865,\ {}_{0.4}p_{68} = 1 - 0.4q_{68} = 1 - 0.4(0.16) = 0.936.$

Finally, ${}_{0.4}p_{[66]+0.6} = p_{[66]}/{}_{0.6}p_{[66]} = 0.88/(1 - 0.6 \times 0.12) = 0.948276.$

So, the answer is ${}_{1.8}p_{[66]+0.6} = 0.76776.$

Method 2: ${}_{1.8}p_{[66]+0.6} = l_{[66]+2.4}/l_{[66]+0.6}$

Without loss of generality, let $l_{[66]} = 100$. Then we have:

$l_{[66]+1} = 88,\ l_{[66]+2} = 88 \times 0.865 = 76.12,\ l_{[66]+2} = 76.12 \times 0.84 = 63.9408.$

So, $l_{[66]+2.4} = 0.4 \times 63.9408 + 0.6 \times 76.12 = 71.24832,$

$l_{[66]+0.4} = 0.6 \times 88 + 0.4 \times 100 = 92.8,$ and ${}_{1.8}p_{[66]+0.6} = 0.767762.$

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