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Study Manual for Exam FAM-S

2nd Edition

Sam A. Broverman, PhD, ASA

An SOA Exam

ACTEX Learning

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2nd Edition

Sam A. Broverman, PhD, ASA

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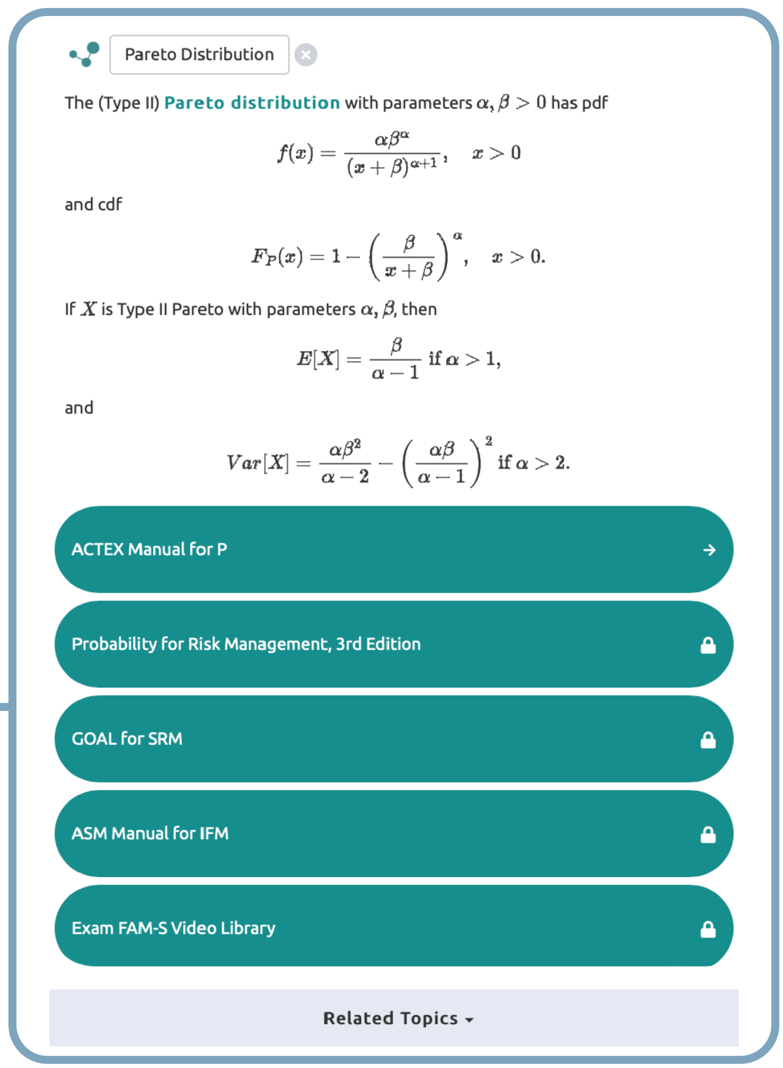

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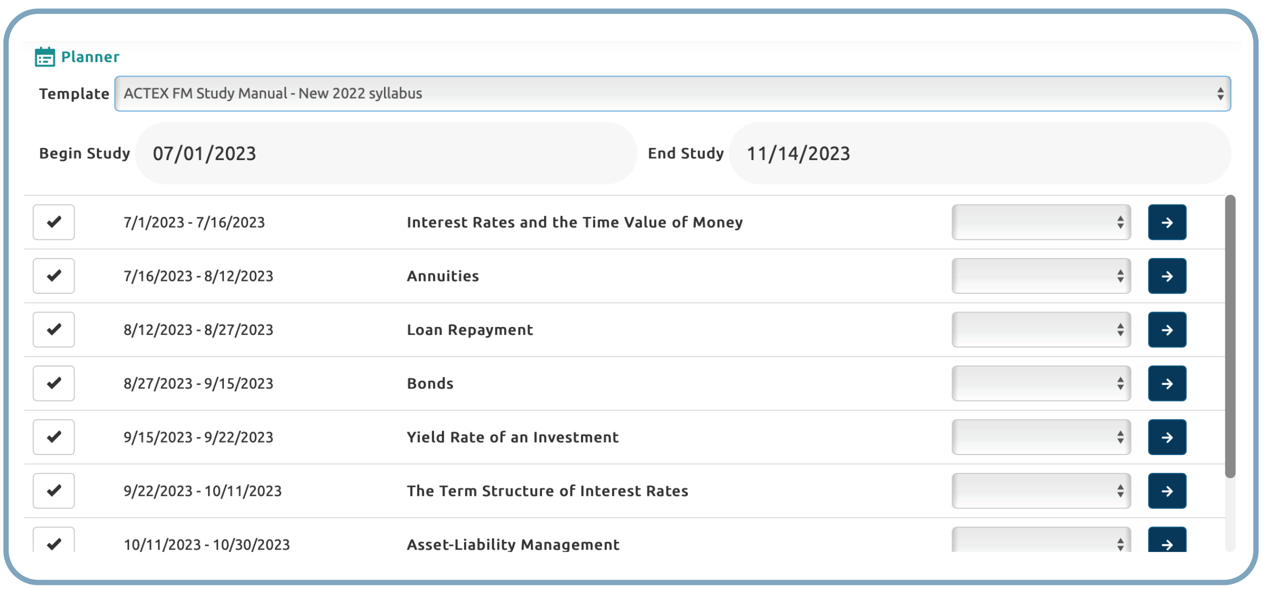

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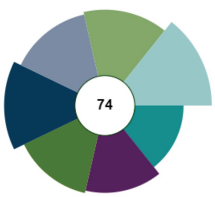

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See key areas where you can improve.

A. Multiple State Models — A1. Multiple State Probabilities

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03/24/2023	03/24/2023	03/24/2023	Quiz	G. Pension Plans and Retirement Benefits	15	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	C1. Joint Life/Last Survivor Probabilities, C2. Contingent Probabilities, C3. Expectation and Variance, C6. ML Models with Transition Intensities, C4. ML Models: Premiums and Reserves (Discrete), C5. ML Models: Premiums and Reserves (Continuous)	15	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	F. Embedded Options and Equity-Linked Insurance	20	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	D. Profit Analysis	10	Complete	Review
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Quickly return to previous sessions.

Contents

INTRODUCTORY COMMENTS

The FAM exam is divided almost equally into FAM-S and FAM-L topics. This study manual is designed to help in the preparation for the FAM-S part of the Society of Actuaries FAM Exam.

The first part of this manual consists of a summary of notes, illustrative examples and problem sets with detailed solutions. The second part consists of 5 practice exams. The SOA exam syllabus for the FAM exam indicates that the exam is 3.5 hours in length with 34 multiple choice questions. The practice exams in this manual each have 17 questions, reflecting the fact that FAM-S is 50% of the full FAM exam. The appropriate time for the 17 question FAM-S practice exams in this manual is one hour and forty-five minutes.

The level of difficulty of the practice exam questions has been designed to be similar to those on past exams covering the same topics. The practice exam questions are not from old SOA exams.

I have attempted to be thorough in the coverage of the topics upon which the exam is based, and consistent with the notation and content of the official references. I have been, perhaps, more thorough than necessary on reviewing maximum likelihood estimation.

Because of the time constraint on the exam, a crucial aspect of exam taking is the ability to work quickly. I believe that working through many problems and examples is a good way to build up the speed at which you work. It can also be worthwhile to work through problems that you have been done before, as this helps to reinforce familiarity, understanding and confidence. Working many problems will also help in being able to more quickly identify topic and question types. I have attempted, wherever possible, to emphasize shortcuts and efficient and systematic ways of setting up solutions. There are also occasional comments on interpretation of the language used in some exam questions. While the focus of the study manual is on exam preparation, from time to time there will be comments on underlying theory in places that I feel those comments may provide useful insight into a topic.

The notes and examples are divided into 31 sections of varying lengths, with some suggested time frames for covering the material. There are almost 180 examples in the notes and over 440 exercises in the problem sets, all with detailed solutions. The 5 practice exams have 17 questions each, also with detailed solutions. Some of the examples and exercises are taken from previous SOA exams. Some of the problem set exercises are more in depth than actual exam questions, but the practice exam questions have been created in an attempt to replicate the level of depth and difficulty of actual exam questions. In total there are almost 700 examples/problems/sample exam questions with detailed solutions. ACTEX gratefully acknowledges the SOA for allowing the use of their exam problems in this study manual.

I suggest that you work through the study manual by studying a section of notes and then attempting the exercises in the problem set that follows that section. The order of the sections of notes is the order that I recommend in covering the material, although the material on short-term insurance pricing and reserving in Sections 27 to 30 and option pricing in Section 31 is independent of the other material on the exam. The order of topics in this manual is not the same as the order presented on the exam syllabus.

It has been my intention to make this study manual self-contained and comprehensive, however it is important to be familiar with original reference material on all topics.

While the ability to derive formulas used on the exam is usually not the focus of an exam question, it is useful in enhancing the understanding of the material and may be helpful in memorizing formulas. There may be an occasional reference in the review notes to a derivation, but you are encouraged to review the official reference material for more detail on formula derivations.

In order for the review notes in this study manual to be most effective, you should have some background at the junior or senior college level in probability and statistics. It will be assumed that you are reasonably familiar with differential and integral calculus. The prerequisite concepts to modeling and model estimation are reviewed in this study manual. The study manual begins with a detailed review of probability distribution concepts such as distribution function, hazard rate, expectation and variance. Of the various calculators that are allowed for use on the exam, I am most familiar with the BA II PLUS. It has several easily accessible memories. The TI-30X IIS has the advantage of a multi-line display. Both have the functionality needed for the exam.

There is a set of tables that has been provided with the exam in past sittings. These tables consist of some detailed description of a number of probability distributions along with tables for the standard normal and chi-squared distributions. The tables can be downloaded from the SOA website www.soa.org.

If you have any questions, comments, criticisms or compliments regarding this study manual, please contact the publisher ACTEX, or you may contact me directly at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you would bring them to my attention. ACTEX will be maintaining a website for errata that can be accessed from https://actexlearning.com/errata. It is my sincere hope that you find this study manual helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

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Section 7

Mixture Of Two Distributions

The material in this section relates to Section 4.2.3 of "Loss Models". The suggested time frame for this section is 2 hours. The topic of distribution mixtures is not mentioned in the learning objectives for the FAM exam but all of Chapter 4 of "Loss Models" is listed in the reference reading.

7.1 Mixture of Two Distributions

We begin with a formal algebraic definition of how a mixture of two distributions is constructed, and later we will look at how a mixture distribution is described by general reasoning.

Given random variables X_1 and X_2, with pdf's or pf's $f_{X_1}(x)$ and $f_{X_2}(x)$, and given $0 < a < 1$, we construct the random variable Y with pdf

$$\boldsymbol{f_Y(y) = a \times f_{X_1}(y) + (1-a) \times f_{X_2}(y)} \tag{7.1}$$

Y is called a **mixture distribution** or a two-point mixture of the distributions of X_1 and X_2.

The two-point mixture random variable Y can also be defined in terms of the cdf,

$$\boldsymbol{F_Y(y) = a \times F_{X_1}(y) + (1-a) \times F_{X_2}(y)} \tag{7.2}$$

X_1 and X_2 are the **component distributions** of the mixture, and the factors a and $1-a$ are referred to as **mixing weights**. It is important to understand that we are **not adding** aX_1 and $(1-a)X_2$, Y **is not** a sum of random variables. Y is defined in terms of a pdf (or cdf) that is a weighted average of the pdf's (or cdf's) of X_1 and X_2. We are adding af_{X_1} and $(1-a)f_{X_2}$ to get f_Y.

Example 7.1. As a simple illustration of a mixture distribution, consider two bowls. Bowl A has 5 balls with the number 1 on them and 5 balls with the number 2 on them, and bowl B has 3 balls with the number 1 and 7 balls with the number 2. Let X_1 denote the number on a ball randomly chosen from bowl A, and let X_2 denote the number on a ball randomly chosen from bowl B. The probability functions of X_1 and X_2 are
$f_{X_1}(1) = f_{X_1}(2) = 0.5$ and $f_{X_2}(1) = 0.3$, $f_{X_2}(2) = 0.7$.

Suppose we create the mixture distribution with mixing weights $a = 0.5$ and $1-a = 0.5$. The mixture distribution Y has probability function $f_Y(1) = 0.5 \times 0.5 + 0.5 \times 0.3 = 0.4$, $f_Y(2) = 0.5 \times 0.5 + 0.5 \times 0.7 = 0.6$.

Note that the outcomes (ball numbers) of the mixture distribution Y come from the possible outcomes of the component distributions X_1 and X_2. □

An alternative way of looking at this mixture distribution is by means of conditioning on a "parameter". This will be important when we look at continuous mixing. The parameter approach to describe the mixture distribution in Example 7.1 is as follows.

Suppose that a fair coin is tossed. If the toss is a head, a ball is chosen at random from bowl A and if the toss is a tail, a ball is chosen at random from bowl B. We define the random variable Z to be the number on the ball. We will see that Z has the same distribution as the mixture distribution labeled Y above.

The random variable Z can be interpreted as follows. Consider the 2-point random variable Θ, for which $\Theta =$Bowl A if the coin toss is a head, and $\Theta =$Bowl B if the toss is a tail. Then $P(\Theta = A) = P(\Theta = B) = .5$ (since the coin is fair). Θ is used to indicate which bowl the ball will be chosen from depending on the outcome of the coin toss.

If the toss is a head, the bowl is A, and then Z has the X_1 distribution for the number on the ball, so $f_{X_1}(1) = P(Z = 1|\Theta = A) = 0.5$ and $f_{X_1}(2) = P(Z = 2|\Theta = A) = 0.5$. In a similar way, if the toss is a tail, the bowl is B, and then Z has the X_2 distribution for the number on the ball, so $f_{X_2}(1) = P(Z = 1|\Theta = B) = 0.3$ and $f_{X_2}(2) = P(Z = 2|\Theta = B) = 0.7$.
Z is described as a combination of two conditional distributions based on the parameter Θ.

To find the overall, or unconditional distribution of Z, we use some basic rules of probability. Since Θ must be A or B, we can think of bowl B as the "complement" of bowl A, and then

$$\begin{aligned}P(Z = 1) &= P[(Z = 1) \cap (\Theta = A)] + P[(Z = 1) \cap (\Theta = B)]\\ &= P(Z = 1|\Theta = A) \times P(\Theta = A) + P(Z = 1|\Theta = B) \times P(\Theta = B)\\ &= 0.5 \times 0.5 + 0.5 \times 0.3 = 0.4\end{aligned}$$

We have used the rule $P(C) = P[C \cap D] + P[C \cap D'] = P(C|D) \times P(D) + P(C|D') \times P(D')$.

This shows that the distribution of Z is the same as the mixture distribution Y in Example 7.1. The mixing weights for the two bowls are the probabilities of the coin indicating bowl A or bowl B.

Language used on exam questions that identifies a mixture distribution
There is some typical language that is used in exam questions that indicates that a mixture of distributions is being considered. This language will be illustrated using the distributions involved in the bowl example.

Suppose that we are told that there are two types of individuals. Type A individuals have a mortality probability of .5 (and survival probability of .5) in the coming year, and Type B individuals have a mortality probability of .3 in the coming year. In a large group of these individuals, 50% are Type A and 50% are Type B. An individual is chosen at random from the group. We want to find this randomly chosen individual's mortality probability.

We can use the usual rules of conditional probability to formulate this probability, just as above:

$$\begin{aligned}P(\text{dying this year}) &= P(\text{dying} \cap \text{Type A}) + P(\text{dying} \cap \text{Type B})\\ &= P(\text{dying}|\text{Type A}) \times P(\text{Type A}) + P(\text{dying} \cap \text{Type B}) \times P(\text{Type B})\\ &= 0.5 \times 0.5 + 0.5 \times 0.3 = 0.4.\end{aligned}$$

This is exactly $f_Y(1) = f_{X_1}(1) \times a + f_{X_2}(1) \times (1 - a)$ where "1" corresponds to the event of dying within the year. Note that the phrase "50% are Type A" is interpreted as meaning that if an individual is chosen from the large group, the probability of being Type A is 0.5. This is language that is often used in exam questions to indicate a mixture.

Mixture of a discrete and a continuous distribution

In Section 2 of this study guide we looked at "mixed distributions". In that section, a mixed distribution referred to a random variable that was continuous on part of its probability space and also had one or more discrete points. The concept of mixed distribution just introduced in this section can be used to describe the Section 2 type of mixed distribution. The following example uses the example from Section 2 to illustrate this.

Example 7.2. Suppose that X is defined in the following piecewise way: X has a discrete point of probability of .5 at $X = 0$, and on the interval (0,1) X is a continuous random variable with density function $f(x) = x$ for $0 < x < 1$, and X has no density or probability elsewhere. Show that this random variable can be described as a mixture of two distributions with appropriate definitions for component distributions X_1, X_2 and mixing weights a and $1 - a$.

Solution.

Let $X_1 = 0$ be a constant (not actually a random variable, but is sometimes called a degenerate random variable), so that $f_{X_1}(0) = P(X_1 = 0) = 1$ and $f_{X_1}(x) = 0$ for $x \neq 0$.

Let X_2 be continuous on (0,1) with pdf $f_{X_2}(x) = 2x$.

With mixing weights of $a = 0.5$ and $1 - a = 0.5$, using the definition of the mixture of two distributions, we have the mixed random variable Y satisfying

$f_Y(0) = a \times f_{X_1}(0) + (1 - a) \times f_{X_2}(0) = (0.5 \times 1) + 0 = 0.5$, and
$f_Y(x) = a \times f_{X_1}(x) + (1 - a) \times f_{X_2}(x) = 0.5 \times 0 + 0.5 \times 2x = x$ for $0 < x < 1$.
Y has the same distribution as the original X. □

We should be a little careful about the situation in Example 7.2. When we are mixing discrete probability points with a continuous density, for any particular discrete point we always assign a probability of 0 at that point for any continuous component distribution. For instance, in Example 7.2, if X_1 was at the single point 0.4, then $f_{X_1}(0.4) = 1$ and $f_{X_1}(x) = 0$ elsewhere, and for the mixture distribution, then $f_Y(0.4) = 0.5 \times 1 = 0.5$ (we do not add $0.5 \times f_{X_2}(0.4)$).

Some important relationships for mixture distributions

We have already seen the defining relationships

$$\boldsymbol{f_Y(y) = af_{X_1}(y) + (1 - a)f_{X_2}(y)}$$

and

$$\boldsymbol{F_Y(y) = aF_{X_1}(y) + (1 - a)F_{X_2}(y)}.$$

We can interpret these relationships by saying that the pdf and cdf of Y are weighted averages of the component pdf's and cdf's. This weighted average interpretation can be applied to a number of other distribution related quantities.

- if C is any event related to Y, then $P(C) = a \times P_{X_1}(C) + (1-a) \times P_{X_2}(C)$; (7.3)
  $P_{X_1}(C)$ is the event probability based on random variable X_1, and the same for X_2
- If g is any function (that doesn't involve the parameters of Y), then (7.4)
  $E[g(Y)] = a \times E[g(X_1)] + (1-a) \times E[g(X_2)]$

The justification for this relationship follows from the form of the mixed density;

$$\begin{aligned} E[g(Y)] &= \int g(y)\, f_Y(y)\, dy = \int g(y) \times [a \times f_{X_1}(y) + (1-a) \times f_{X_2}(y)]\, dy \\ &= a \times \int g(y) \times f_{X_1}(y)\, dy + (1-a) \times \int g(y) \times f_{X_2}(y)\, dy \\ &= a \times E[g(X_1)] + (1-a) \times E[g(X_2)] \end{aligned}$$

Some of the important examples of these weighted average relationships are:

Interval Probability: the event C is $c < Y \le d$

$$\boldsymbol{P(c < Y \le d) = a \times P(c < X_1 \le d) + (1-a) \times P(c < X_2 \le d)} \tag{7.5}$$

kth moment of Y: the function g is $g(Y) = Y^k$

$$\boldsymbol{E[Y^k] = a \times E[X_1^k] + (1-a) \times E[X_2^k]} \tag{7.6}$$

7.2 Formulating a Mixture Distribution as a Combination of Conditional Distributions

The relationships above can also be formulated as applications of probability rules involving conditional probability and conditional expectation.

(i) For any random variable Y and event D,

$$\boldsymbol{f_Y(y) = f(y|D) \times P(D) + f(y|D') \times P(D')} \tag{7.7}$$

(ii) For any events C and D,

$$\boldsymbol{P[C] = P[C \cap D] + P[C \cap D'] = P[C|D] \times P[D] + P[C|D'] \times P[D'].} \tag{7.8}$$

(iii) For any random variable Y and any event D,

$$\boldsymbol{E[Y] = E[Y|D] \times P[D] + E[Y|D'] \times P[D']} \tag{7.9}$$

We can define the random variable Θ to be 1 or 2, indicating whether or not event D has occurred, so $\Theta = 1 \equiv D$ has occurred, and $\Theta = 2 \equiv D'$ has not occurred.

We can think of X_1 as the conditional distribution of Y given $\Theta = 1$ (event D) and we can think of X_2 as the conditional distribution of Y given $\Theta = 2$ (event D'), so that

$$f(y|D) = f(y|\Theta = 1) = f_{X_1}(y) \text{ and } f(y|D') = f(y|\Theta = 2) = f_{X_2}(y).$$

The cdf, pdf and expectation of Y are adaptations of the conditional probability rules above. For instance, the mean of the mixed distribution is the "mixture of the means" of the component distributions.

$$E[Y] = E[Y|\Theta = 1] \times P[\Theta = 1] + E[Y|\Theta = 2] \times P[\Theta = 2] = E[X_1] \times a + E[X_2] \times (1-a).$$

Example 7.3. A collection of insurance policies consists of two types. 25% of policies are Type 1 and 75% of policies are Type 2. For a policy of Type 1, the loss amount per year follows an exponential distribution with mean 200, and for a policy of Type 2, the loss amount per year follows a Pareto distribution with parameters $\alpha = 3$ and $\theta = 200$. For a policy chosen at random from the entire collection of both types of policies, find the probability that the annual loss will be less than 100, and find the average loss.

Solution.

The two types of losses are the random variables X_1 and X_2.

X_1 has an exponential distribution with mean 200, so it has cdf $F_{X_1}(x) = 1 - e^{-x/200}$.

X_2 has cdf $F_{X_2}(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^{\alpha} = 1 - \left(\frac{200}{x+200}\right)^{3}$.

These are the component distributions. The mixing weights are the proportions of policies of each type, so that $a = 0.25$ (the proportion of Type 1 policies) and $1 - a = 0.75$. The loss random variable Y of the randomly chosen policy has a distribution that is the mixture of X_1 and X_2. In this context, "randomly chosen" is taken to mean that the probabilities of choosing a Type 1 or Type 2 policy are the proportions of those policy types in the full collection of policies.

The cdf of Y is

$$\begin{aligned} F_Y(y) &= a \times F_{X_1}(y) + (1-a) \times F_{X_2}(y) \\ &= 0.25 \times (1 - e^{-y/200}) + 0.75 \times \left(1 - \left(\frac{200}{y+200}\right)^3\right). \end{aligned}$$

Then, $F_Y(100) = 0.626$.

The mean of Y is $0.25 \times E[X_1] + 0.75 \times E[X_2] = 0.25 \times 200 + 0.75 \times \frac{200}{3-1} = 125$. □

7.3 The Variance of a Mixed Distribution IS (usually) NOT the Weighted Average of the Variances

For a mixture of two random variables, the weighted average pattern applies to most quantities for the mixed distribution.

One important exception to this rule is the formulation of the **variance of the mixture**. If Y is any random variable, then we can formulate the variance of Y as $Var[Y] = E[Y^2] - (E[Y])^2$.

If Y is the mixture of X_1 and X_2 with mixing weights a and $1-a$, then we know that $E[Y] = a \times E[X_1] + (1-a) \times E[X_2]$ and $E[Y^2] = a \times E[X_1^2] + (1-a) \times E[X_2^2]$.

It is tempting to guess that $Var[Y]$ is the weighted average of $Var[X_1]$ and $Var[X_2]$.

This is **<u>not true</u>** in general (we will see a special case later for which it is true).

Example 7.4. Find the variance of the loss on the policy chosen at random in Example 7.3, and compare it to the weighted average of the variances of the two component loss distributions.

Solution.

The mean of Y is $0.25 \times E[X_1] + 0.75 \times E[X_2] = 0.25 \times 200 + 0.75 \times \frac{200}{3-1} = 125$,

and the second moment of Y is

$0.25 \times E[X_1^2] + 0.75 \times E[X_2^2] = 0.25 \times 2 \times 200^2 + 0.75 \times \frac{200^2 \times 2}{(3-1)(3-2)} = 50{,}000$, so

$Var[Y] = E[Y^2] - (E[Y])^2 = 34{,}375$.

$Var[X_1] = 40{,}000$ (variance of an exponential distribution is the square of the mean).

$Var[X_2] = \frac{200^2 \times 2}{(3-1)(3-2)} - 100^2 = 30{,}000$.

The weighted average of the two variances is $0.25 \times 40{,}000 + 0.75 \times 30{,}000 = 32{,}500$, which is not the variance of Y. □

Weighted average does not apply to percentiles in a mixture distribution
In Example 7.3, the 50th percentile of X_1 is m_1, the solution of the equation
$F_{X_1}(m_1) = 1 - e^{-m_1/200} = 0.5$, so that $m_1 = 138.63$.

The 50th percentile of X_2 is m_2, the solution of $F_{X_2}(m_2) = 1 - \left(\frac{200}{m_2+200}\right)^3 = 0.5$,
so that $m_2 = 51.98$.

The weighted average of the two 50th percentiles is $0.25 \times 138.63 + 0.75 \times 51.98 = 73.64$.
The cdf of the mixed distribution Y is $F_Y(y) = 0.25 \times F_{X_1}(y) + 0.75 \times F_{X_2}(y)$, and
$F_Y(73.64) = 0.53$, so 73.64 is not the 50th percentile of Y.

To find the 50th percentile, say m, of the mixed distribution in Example 7.3, we must solve the equation

$$\begin{aligned} .50 &= F_Y(m) = 0.25 \times F_{X_1}(m) + 0.75 \times F_{X_2}(m) \\ &= 0.25 \times (1 - e^{-m/200}) + 0.75 \times \left(1 - \left(\frac{200}{m+200}\right)^3\right). \end{aligned}$$

There is no algebraic solution, and m would have to be found by a numerical approximation method. The 50th percentile turns out to be about 65.7.

Section 7 Problem Set

Mixture Of Two Distributions

1. A portfolio of insurance policies is divided into low and high risk policies. 80% of the policies are low risk and the rest are high risk. The annual number of claims on a low risk policy has a Poisson distribution with a mean of .25 and the annual number of claims on a high risk policy has a Poisson distribution with a mean of 2. A policy is randomly chosen from the portfolio.

   (a) Find the mean and variance of the number of annual claims for the policy.

   (b) Find the probability that the policy has at most 1 claim in the coming year.

2. The distribution of a loss, X, is a two-point mixture:

   (i) With probability 0.8, X has a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 100$.

   (ii) With probability 0.2, X has a two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 3000$.

   Calculate $Pr(X \leq 200)$.

   (A) 0.76 (B) 0.79 (C) 0.82 (D) 0.85 (E) 0.88

3. The random variable N has a mixed distribution:

   (i) With probability p, N has a binomial distribution with $q = 0.5$ and $m = 2$.

   (ii) With probability $1 - p$, N has a binomial distribution with $q = 0.5$ and $m = 4$.

   Which of the following is a correct expression for Prob$(N = 2)$?

   (A) $0.125p^2$
   (B) $0.375 + 0.125p$
   (C) $0.375 + 0.125p^2$
   (D) $0.375 - 0.125p^2$
   (E) $0.375 - 0.125p$

4. Y is a mixture of two exponential distributions, $f_Y(y) = \frac{1}{2}e^{-y} + \frac{1}{6}e^{-y/3}$.
   The random variable Z defined by the equation $Z = 2Y$.
   Z is a mixture of two exponentials. The means of those two exponential distributions are

   (A) 1 and 3
   (B) 1 and 6
   (C) 2 and 3
   (D) 2 and 6
   (E) 3 and 6

5. X_1 has a uniform distribution on the interval (0,1000) and X_2 has a uniform distribution on the interval (0,2000). Y is defined as a mixture of X_1 and X_2 with mixing weights of .5 for each mixture component. Find the pdf, cdf and median (50th percentile) of Y.

6. A population of people aged 50 consists of twice as many non-smokers as smokers. Non-smokers at age 50 have a mortality probability of .1 and smokers at age 50 have a mortality probability of .2. Two 50-year old individuals are chosen at random from the population.

   (a) Find the probability that at least one of them dies before age 51.

   (b) Suppose that the mortality probabilities for smokers and non-smokers remain the same at age 51. Find the mortality probability of a randomly chosen survivor at age 51 in this population.

7. Y is the mixture of an exponential random variable with mean 1 and mixing weight $\frac{2}{3}$, and an exponential distribution with mean 2 and mixing weight $\frac{1}{3}$.

   Find the pdf, cdf, mean, variance and 90th percentile of Y.

8. Which of the following statements are true?

   I. A mixture of two different exponentials with the mixture having a mean 2 has a heavier right tail than a single exponential distribution with mean 2.

   II. If $f_Y(y) = a_1 \times f_{X_1}(y) + a_2 \times f_{X_2}(y)$, where $0 < a_1 < 1$ and $0 < a_2 < 1$, then $e_Y(d) = a_1 \times e_{X_1}(d) + a_2 \times e_{X_2}(d)$.

9. You are given two independent estimates of an unknown quantity, μ:

   (i) Estimate A: $E(\mu_A)$= 1000 and $\sigma(\mu_A)$= 400

   (ii) Estimate B: $E(\mu_B)$= 1200 and $\sigma(\mu_B)$= 200

   Estimate C is a weighted average of the two estimates A and B, such that:

   $$\mu_C = w \times \mu_A + (1 - w) \times \mu_B$$

   Determine the value of w that minimizes $\sigma(\mu_C)$.

   (A) 0 (B) 1/5 (C) 1/4 (D) 1/3 (E) 1/2

10. You are given the claim count data for which the sample mean is roughly equal to the sample variance. Thus you would like to use a claim count model that has its mean equal to its variance. An obvious choice is the Poisson distribution. Determine which of the following models may also be appropriate.

    (A) A mixture of two binomial distributions with different means.

    (B) A mixture of two Poisson distributions with different means.

    (C) A mixture of two negative binomial distributions with different means.

    (D) None of A, B, or C

    (E) All of A, B, and C

11. Losses come from an equally weighted mixture of an exponential distribution with mean m_1, and an exponential distribution with mean m_2. Determine the least upper bound for the coefficient of variation of this distribution.

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2 (E) $\sqrt{5}$

12. X has the following distribution : $Pr[X = 0] = 0.4$, $Pr[X = 1] = 0.6$.
The distribution of Y is conditional on the value of X: if $X = 0$, then the distribution of Y is $Pr[Y = 0] = 0.6$, $Pr[Y = 1] = 0.2$, $Pr[Y = 2] = 0.2$, and if $X = 1$, then the distribution of Y is $Pr[Y = 0] = 0.2$,$Pr[Y = 1] = 0.3$, $Pr[Y = 2] = 0.5$.
Z is the sum of Y independent normal random variables, each with mean and variance 2.

What is $Var[Z]$?

(A) 5.0 (B) 6.0 (C) 7.0 (D) 8.0 (E) 9.0

13. Subway trains arrive at your station at a Poisson rate of 20 per hour. 25% of the trains are express and 75% are local. The types and number of trains arriving are independent. An express gets you to work in 16 minutes and a local gets you there in 28 minutes. You always take the first train to arrive. Your co-worker always takes the first express. You are both waiting at the same station.

Calculate the conditional probability that you arrive at work before your co-worker, given that a local arrives first.

(A) 37% (B) 40% (C) 43% (D) 46% (E) 49%

Section 7 Problem Set Solutions

1. (a) Y is the mixture of two Poisson distributions.
$E[Y] = 0.8 \times E[X_L] + 0.2 \times E[X_H] = 0.8 \times 0.250 + 0.2 \times 2 = 0.6$.
To find the second moment of Y, we recall that for a Poisson distribution with mean λ, the variance is also λ. Since $Var[X] = E[X^2] - (E[X])^2$, it follows that
$E[X^2] = Var[X] + (E[X])^2 = \lambda + \lambda^2$ for a Poisson distribution.

Then, $E[Y^2] = 0.8 \times E[X_L^2] + 0.2 \times E[X_H^2] = 0.8 \times (0.25 + 0.25^2) + 0.2 \times (2 + 2^2) = 1.45$.
Then $Var[Y] = 1.45 - (0.6)^2 = 1.09$.

(b) $P(Y \le 1) = 0.8 \times P(X_1 \le 1) + 0.2 \times P(X_2 \le 1)$
$= 0.8 \times [e^{-0.25} + 0.25e^{-0.25}] + 0.2 \times [e^{-2} + 2e^{-2}] = 0.860$.

2. The probability is a mixture of the probabilities for the two components.
$P(X \le 200) = 0.8 \times P(X_1 \le 200) + 0.2 \times P(X_2 \le 200)$, where X_1 and X_2 are the two Pareto distributions.

$P(X_1 \le 200) = 1 - \left(\frac{100}{200+100}\right)^2 = .8889$, and
$P(X_2 \le 200) = 1 - \left(\frac{3000}{200+3000}\right)^4 = .2275$.
$P(X \le 200) = (0.8 \times 0.8889) + (0.2 \times 0.2275) = 0.757$.

Answer A

3. $P(N = 2) = p \times P(N_1 = 2) + (1 - p) \times P(N_2 = 2)$
$= p \times (0.5)^2 + (1 - p) \times 6 \times (0.5)^4 = 0.375 - 0.125p$.

We have used the binomial probabilities of the form $\binom{m}{k} q^k (1 - q)^{m-k}$.

Answer E

4. $f_Y(y) = \frac{1}{2} \times e^{-y} + \frac{1}{2} \times \frac{1}{3} e^{-y/3}$. $F_Y(y) = \frac{1}{2} \times (1 - e^{-y}) + \frac{1}{2} \times (1 - e^{-y/3})$.
$F_Z(z) = F_Y(\frac{z}{2}) = \frac{1}{2} \times (1 - e^{-z/2}) + \frac{1}{2} \times (1 - e^{-z/6})$.
Z is a mixture of two exponentials with means 2 and 6.

Answer D

5. $f_Y(y) = 0.5 \times f_{X_1}(y) + 0.5 \times f_{X_2}(y)$

$$= \begin{cases} 0.5 \times 0.001 + 0.5 \times 0.0005 = 0.00075 & \text{if } 0 < y < 1000 \\ 0.5 \times 0.0005 = 0.00025 & \text{if } 1000 \le y < 2000 \end{cases}$$

$F_Y(y) = 0.5 \times F_{X_1}(y) + 0.5 \times F_{X_2}(y)$

$$= \begin{cases} 0.5 \times 0.001x + 0.5 \times 0.0005x = 0.00075x & \text{if } 0 < y < 1000 \\ 0.5 \times 1 + 0.5 \times 0.0005x = 0.5 + 0.00025x & \text{if } 1000 \le y < 2000 \end{cases}$$

The median of Y, m, must satisfy the equation $F_Y(m) = 0.5$.
We see that at $y = 1000$, $F_Y(1000) = 0.75$. Therefore, $m < 1000$, so that
$F_Y(m) = 0.00075m = 0.5$. Therefore, $m = 666.67$.
Note that the mean of Y is $0.5 \times 500 + 0.5 \times 1000 = 750$.

6. For a randomly chosen individual at age 50, the mortality probability is the mixture of the mortality probabilities for non-smokers and smokers. This is $q = \frac{2}{3} \times 0.1 + \frac{1}{3} \times 0.2 = \frac{2}{15}$.

   The probability that both of two independent 50-year old individuals survive the year is $(1 - \frac{2}{15})^2 = 0.7511$. The probability at least one of them dies by age 51 is $1 - 0.7511 = 0.25$.

   Suppose that there are 1000 non-smokers and 500 smokers at age 50.
   The expected number of surviving non-smokers at age 51 is $1000 \times 0.9 = 900$, and the number of surviving smokers is $500 \times 0.8 = 400$.

   A randomly chosen survivor at age 51 has a $\frac{9}{13}$ chance of being a non-smoker and a $\frac{4}{13}$ chance of being a smoker.

   The mortality probability at age 51 for the randomly chosen survivor is
   $\frac{9}{13} \times 0.1 + \frac{4}{13} \times 0.2 = .131$.

7. $f_Y(y) = \frac{2}{3} \times e^{-y} + \frac{1}{3} \times \frac{1}{2} e^{-y/2}$.

   $F_Y(y) = \frac{2}{3} \times (1 - e^{-y}) + \frac{1}{3} \times (1 - e^{-y/2})$.

   $E[Y] = \frac{2}{3} \times 1 + \frac{1}{3} \times 2 = \frac{4}{3}$.

   $E[Y^2] = \frac{2}{3} \times (2 \times 1^2) + \frac{1}{3} \times (2 \times 2^2) = 4 \to Var[Y] = 4 - (\frac{4}{3})^2 = \frac{20}{9}$.

   The 90th percentile of Y is c, which must satisfy the equation
   $F_Y(c) = \frac{2}{3} \times (1 - e^{-c}) + \frac{1}{3} \times (1 - e^{-c/2}) = 0.9$.

   If we let $r = e^{-c/2}$, then this becomes a quadratic equation in r,
   $\frac{2}{3} \times (1 - r^2) + \frac{1}{3} \times (1 - r) = 0.9$, or equivalently, $2r^2 + r - .3 = 0$.

   Solving for r results in $r = .21$ or $r = -0.71$.

   We ignore the negative root, since $r = e^{-c/2}$ must be positive.

   Then $c = -2 \ln r = -2 \ln .21 = 3.12$ is the 90th percentile of Y.

8. I. If $f_Y(y) = a_1 \times f_{X_1}(y) + a_2 \times f_{X_2}(y) = a_1 \times \frac{1}{\theta_1} e^{-y/\theta_1} + a_2 \times \frac{1}{\theta_2} e^{-y/\theta_2}$,
   and $a_1\theta_1 + a_2\theta_2 = 2$. Then $\theta_1 < 2 < \theta_2$ (or vice-versa).

   For an exponential distribution with mean 2, we have $f_Z(y) = \frac{1}{2} e^{-y/2}$
   $\frac{f_Y(y)}{f_Z(y)} = \frac{a_1 \times \frac{1}{\theta_1} e^{-y/\theta_1} + a_2 \times \frac{1}{\theta_2} e^{-y/\theta_2}}{\frac{1}{2} e^{-y/2}} = \frac{2a_1}{\theta_1} e^{y(\frac{1}{2} - \frac{1}{\theta_1})} + \frac{2a_2}{\theta_2} e^{y(\frac{1}{2} - \frac{1}{\theta_2})}$.

   Since $\theta_1 < 2 < \theta_2$, it follows that $\frac{1}{\theta_2} < \frac{1}{2} < \frac{1}{\theta_1}$. Therefore, $\frac{1}{2} - \frac{1}{\theta_2} > 0$
   so that $\lim_{y \to \infty} \frac{f_Y(y)}{f_Z(y)} = \lim_{y \to \infty} \frac{2a_1}{\theta_1} e^{y(\frac{1}{2} - \frac{1}{\theta_1})} + \frac{2a_2}{\theta_2} e^{y(\frac{1}{2} - \frac{1}{\theta_2})} = 0 + \infty$. True.

   II. Suppose that $f_Y(y) = \frac{1}{2} e^{-y} + \frac{1}{6} e^{-y/3}$. Then $S_Y(y) = \frac{1}{2} e^{-y} + \frac{1}{2} e^{-y/3}$,
   so that Y is mixture of exponentials with means of 1 and 3, with equal mixing weights of $\frac{1}{2}$.
   $e_Y(d) = \frac{\int_d^\infty S(y)\, dy}{S(d)} = \frac{\frac{1}{2} e^{-d} + \frac{3}{2} e^{-d/3}}{\frac{1}{2} e^{-d} + \frac{1}{2} e^{-d/3}}$.
   $e_{X_1}(d) = 1$, $e_{X_3}(d) = 3 \;\to\; \frac{1}{2} \times e_{X_1}(d) + \frac{1}{2} \times e_{X_3}(d) = 2$. False.

9. Since the two estimates are independent,

$$Var[\mu_C] = Var[w \times \mu_A + (1-w) \times \mu_B] = w^2 \times Var[\mu_A] + (1-w)^2 \times Var[\mu_B]$$
$$= w^2 \times 400^2 + (1-w)^2 \times 200^2 = 200{,}000w^2 - 80{,}000w + 40{,}000.$$

This is a quadratic expression in w with positive coefficient of w^2. The minimum can be found by differentiating with respect to w and setting equal to 0:
$400{,}000w - 80{,}000 = 0 \;\rightarrow\; w = \frac{1}{5}$ is the value of w that minimizes $Var[\mu_c]$.

Answer B

10. If Y is a mixture of X_1 and X_2 with mixing weights a and $1-a$, we can define the parameter $\Theta = \{1,2\}$, with $P[\Theta = 1] = a\,, P[\Theta = 2] = 1-a$.
Then $E[Y] = E[\,E[Y|\Theta]\,] = E[Y|\Theta = 1] \times P[\Theta = 1] + E[Y|\Theta = 2] \times P[\Theta = 2]$
$= E[X_1] \times a + E[X_2] \times (1-a)$
(this is the usual way the mean of a finite mixture is formulated).

$Var[Y] = E[\,Var[Y|\Theta]\,] + Var[\,E[Y|\Theta]\,]$, where
$E[\,Var[Y|\Theta]\,] = Var[X_1] \times a + Var[X_2] \times (1-a)$ and
$Var[\,E[Y|\Theta]\,] = (E[X_1] - E[X_2])^2 \times a \times (1-a)$.

The last equality follows from the fact that if Z is a two-point random variable

$Z = \begin{cases} u & \text{prob.}\, p \\ v & \text{prob.}\, 1-p \end{cases}$, then $Var[Z] = (u-v)^2 \times p \times (1-p)$; $E[Y|\Theta]$ is

a two-point random variable $E[Y|\Theta] = \begin{cases} E[X_1] & \text{prob.}\, a \\ E[X_2] & \text{prob.}\, 1-a \end{cases}$.

Therefore $Var[Y] = Var[X_1] \times a + Var[X_2] \times (1-a) + (E[X_1] - E[X_2])^2 \times a \times (1-a)$.

B) If X_1,X_2 are Poisson random variables then $E[X_1] = Var[X_1]$ and $E[X_2] = Var[X_2]$, so that $E[Y] = E[X_1] \times a + E[X_2] \times (1-a) = Var[X_1] \times a + Var[X_2] \times (1-a)$.

It follows that
$Var[Y] = Var[X_1] \times a + Var[X_2] \times (1-a) + (E[X_1] - E[X_2])^2 \times a \times (1-a) > E[Y]$.

C) For a negative binomial random variable with parameters r and β, the mean is $r\beta$ and the variance is $r\beta(1+\beta)$, so the variance is larger than the mean. If X_1 and X_2 have negative binomial distributions, the $E[X_1] < Var[X_1]$ and $E[X_2] < Var[X_2]$.

Therefore, $E[Y] = E[X_1] \times a + E[X_2] \times (1-a) < Var[X_1] \times a + Var[X_2] \times (1-a)$, and
$Var[X_1] \times a + Var[X_2] \times (1-a)$
$< Var[X_1] \times a + Var[X_2] \times (1-a) + (E[X_1] - E[X_2])^2 \times a(1-a) = Var[Y]$;
therefore $E[Y] < Var[Y]$.

A) For a binomial random variable with parameters m and q, the mean is mq and the variance is $mq(1-q)$, which is smaller than the mean.

Therefore $E[Y] = E[X_1] \times a + E[X_2] \times (1-a) > Var[X_1] \times a + Var[X_2] \times (1-a)$, and it is possible that when we add $(E[X_1] - E[X_2])^2 \times a \times (1-a)$ to the right side, we get approximate equality.
So it is possible that $E[Y] = Var[Y]$ for a mixture of binomials.

Answer A

11. The mean will be $\frac{1}{2} \times (m_1 + m_2)$ and the variance will be $\frac{1}{2} \times (2m_1^2 + 2m_2^2) - [\frac{1}{2} \times (m_1 + m_2)]^2$.
The coefficient of variation is the ratio of standard deviation to mean.
The square of the coefficient of variation is

$$\begin{aligned}\frac{\frac{1}{2} \times (2m_1^2 + 2m_2^2) - [\frac{1}{2} \times (m_1 + m_2)]^2}{[\frac{1}{2} \times (m_1 + m_2)]^2} &= \frac{3m_1^2 - 2m_1m_2 + 3m_2^2}{(m_1 + m_2)^2} \\ &= \frac{3m_1^2 + 6m_1m_2 + 3m_2^2 - 8m_1m_2}{(m_1 + m_2)^2} \\ &= 3 - \frac{8m_1m_2}{(m_1 + m_2)^2}.\end{aligned}$$

The maximum square of the coefficient of variation is 3, and it occurs at the minimum of $\frac{8m_1m_2}{(m_1+m_2)^2}$, which is 0 (if m_1 or m_2 is 0).
The least upper bound of the coefficient of variation is $\sqrt{3}$.

Answer C

12. $Var[Z] = Var[\,E[Z \mid Y]\,] + E[\,Var[Z \mid Y]\,]$.

$E[Z \mid Y] = 2 \times Y$ (a sum of Y independent normal random variables each with mean 2), and $Var[Z \mid Y] = 2 \times Y$ (a sum of Y independent normal random variables each with a variance of 2).

Thus, $Var[Z] = Var[2 \times Y] + E[2 \times Y] = 4 \times Var[Y] + 2 \times E[Y]$.
$Var[Y] = Var[\,E[Y \mid X]\,] + E[\,Var[Y \mid X]\,]$.
Since $E[Y] = 0.6$ with prob. $0.4\,(X = 0)$ and $E[Y] = 1.3$ with prob. $0.6\,(X = 1)$, it follows that $Var[E[Y \mid X]] = (0.6)^2 \times 0.4 + (1.3)^2 \times 0.6 - (1.02)^2 = 0.1176$.

Also, if $X = 0$, the variance of Y is $1^2 \times 0.2 + 2^2 \times 0.2 - (0.6)^2 = 0.64$,
and if $X = 1$, the variance of Y is $1^2 \times 0.3 + 2^2 \times 0.5 - (1.3)^2 = 0.61$. Thus,
$E[Var[Y \mid X]] = 0.64 \times 0.4 + .61 \times .6 = 0.622$, and so $Var[Y] = 0.1176 + 0.622 = 0.7396$.

Then, the variance of Z is $4 \times 0.7396 + 2 \times 1.02 = 4.9984$. Note that $Var[Y]$ can also be found from $E[Y^2] - (E[Y])^2$. Again, $E[Y] = 1.02$, as above, and now,
$E[Y^2] = E[E[Y^2 \mid X]] = 1 \times 0.4 + 2.3 \times 0.6 = 1.78$, so that $Var[Y] = 0.7396$.

Answer A

13. Given that a local train arrives first, you will get to work 28 minutes after that local train arrives, since you will take it. Your co-worker will wait for first express train. You will get to work before your co-worker if the next express train (after the local) arrives more than 12 minutes after the local. We expect 5 express trains per hour, so the time between express trains is exponentially distributed with a mean of $\frac{1}{5}$ of an hour, or 12 minutes. Because of the lack of memory property of the exponential distribution, since we are given that the next train is local, the time until the next express train after that is exponential with a mean of 12 minutes. Therefore, the probability that after the local, the next express arrives in more than 12 minutes is $P[T > 12]$, where T has an exponential distribution with a mean of 12. This probability is $e^{-12/12} = e^{-1} = 0.368$ (37%).

Answer A

Practice Exam 1

1. The XYZ Insurance Company sells property insurance policies with a deductible of \$5,000, policy limit of \$500,000, and a coinsurance factor of 80%. Let X_i be the individual loss amount of the ith claim and Y_i be the claim payment of the ith claim. Which of the following represents the relationship between X_i and Y_i?

(A) $Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 625{,}000 \\ 500{,}000 & X_i > 625{,}000 \end{cases}$

(B) $Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 4{,}000) & 4{,}000 < X_i \leq 500{,}000 \\ 500{,}000 & X_i > 500{,}000 \end{cases}$

(C) $Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 630{,}000 \\ 500{,}000 & X_i > 630{,}000 \end{cases}$

(D) $Y_i = \begin{cases} 0 & X_i \leq 6{,}250 \\ 0.80\,(X_i - 6{,}250) & 6{,}250 < X_i \leq 631{,}500 \\ 500{,}000 & X_i > 631{,}500 \end{cases}$

(E) $Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 505{,}000 \\ 500{,}000 & X_i > 505{,}000 \end{cases}$

2. A casino has a game that makes payouts at a Poisson rate of 5 per hour and the payout amounts can be any non-negative integer, 1, 2, 3, ... without limit. The probability that any given payout is equal to $i > 0$ is $\frac{1}{2^i}$. Payouts are independent. Calculate the probability that there are no payouts of 1,2, or 3 in a given 20 minute period.

(A) 0.08 (B) 0.13 (C) 0.18 (D) 0.23 (E) 0.28

3. A compound Poisson claim distribution has $\lambda = 3$ and individual claims amounts distributed as follows:

x	$f_X(x)$
5	0.6
10	0.4

Determine the expected cost of an aggregate stop-loss insurance with a deductible of 6.

(A) Less than 15.0

(B) At least 15.0 but less than 15.3

(C) At least 15.3 but less than 15.6

(D) At least 15.6 but less than 15.9

(E) At least 15.9

Use the following information for questions 4 and 5. You are the producer of a television quiz show that gives cash prizes. The number of prizes, N, and prize amounts, X are independent of one another and have the following distributions:

N: $P[N = 1] = 0.8$, $P[N = 2] = 0.2$

X: $P[X = 0] = 0.2$, $P[X = 100] = 0.7$, $P[X = 1000] = 0.1$

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4. Your budget for prizes equals the expected prizes plus $1.25\times$ standard deviation of prizes. Calculate your budget.

(A) 384 (B) 394 (C) 494 (D) 588 (E) 596

5. You buy stop-loss insurance for prizes with a deductible of 200. The cost of insurance includes a 175% relative security load (the relative security load is the percentage of expected payment that is added). Calculate the cost of the insurance.

(A) 204 (B) 227 (C) 245 (D) 273 (E) 357

6. An actuary determines that claim counts follow a negative binomial distribution with unknown β and r. It is also determined that individual claim amounts are independent and identically distributed with mean 700 and variance 1,300. Aggregate losses have a mean of 48,000 and variance 80 million. Calculate the values for β and r.

(A) $\beta = 1.20, r = 57.19$

(B) $\beta = 1.38, r = 49.75$

(C) $\beta = 2.38, r = 28.83$

(D) $\beta = 1{,}663.81, r = 0.04$

(E) $\beta = 1{,}664.81, r = 0.04$

7. Let X_1, X_2, X_3 be independent Poisson random variables with means $\theta, 2\theta$, and 3θ respectively. What is the maximum likelihood estimator of θ based on sample values x_1, x_2, and x_3 from the distributions of X_1, X_2 and X_3, respectively,

- (A) $\dfrac{\bar{x}}{2}$
- (B) $\bar{x}$
- (C) $\dfrac{x_1 + 2x_2 + 3x_3}{6}$
- (D) $\dfrac{3x_1 + 2x_2 + x_3}{6}$
- (E) $\dfrac{6x_1 + 3x_2 + 2x_3}{11}$

8. For a group of policies, you are given:

(i) Losses follow a uniform distribution on the interval $(0, \theta)$, where $\theta > 25$.

(ii) A sample of 20 losses resulted in the following:

Interval	Number of Losses
$x \le 10$	n_1
$10 < x \le 25$	n_2
$x > 25$	n_3

The maximum likelihood estimate of θ can be written in the form $25 + y$. Determine y.

- (A) $\dfrac{25n_1}{n_2 + n_3}$
- (B) $\dfrac{25n_2}{n_1 + n_3}$
- (C) $\dfrac{25n_3}{n_1 + n_2}$
- (D) $\dfrac{25n_1}{n_1 + n_2 + n_3}$
- (E) $\dfrac{25n_2}{n_1 + n_2 + n_3}$

9. The number of claims follows a negative binomial distribution with parameters β and r, where β is unknown and r is known. You wish to estimate β based on n observations, where $\bar{x}$ is the mean of these observations. Determine the maximum likelihood estimate of β.

- (A) $\dfrac{\bar{x}}{r^2}$
- (B) $\dfrac{\bar{x}}{r}$
- (C) $\bar{x}$
- (D) $r\bar{x}$
- (E) $r^2\bar{x}$

10. The following 6 observations are assumed to come from the continuous distribution with pdf $f(x;\theta) = \frac{1}{2}x^2\theta^3 e^{-\theta x}$: 1, 3, 4, 4, 5, 7.

Find the mle of θ.

(A) 0.25 (B) 0.50 (C) 0.75 (D) 1.00 (E) 1.25

11. An analysis of credibility premiums is being done for a particular compound Poisson claims distribution, where the criterion is that the total cost of claims is within 5% of the expected cost of claims with a probability of 90%. It is found that with $n = 60$ exposures (periods) and $\bar{X} = 180.0$, the credibility premium is 189.47. After 20 more exposures (for a total of 80) and revised $\bar{X} = 185$, the credibility premium is 190.88. After 20 more exposures (for a total of 100) the revised $\bar{X}$ is 187.5. Assuming that the manual premium remains unchanged in all cases, and assuming that full credibility has not been reached in any of the cases, find the credibility premium for the 100 exposure case.

(A) 191.5 (B) 192.5 (C) 193.5 (D) 194.5 (E) 196.5

12. For an insurance portfolio, you are given:

(i) For each individual insured, the number of claims follows a Poisson distribution.

(ii) The mean claim count varies by insured, and the distribution of mean claim counts follows gamma distribution.

(iii) For a random sample of 1000 insureds, the observed claim counts are as follows:

Number of Claims, n	0	1	2	3	4	5
Number of Insureds, f_n	512	307	123	41	11	6

$$\sum n f_n = 750, \qquad \sum n^2 f_n = 1494$$

(iv) Claim sizes follow a Pareto distribution with mean 1500 and variance 6,750,000.

(v) Claim sizes and claim counts are independent.

(vi) The full credibility standard is to be within 5% of the expected aggregate loss 95% of the time.

Determine the minimum number of insureds needed for the aggregate loss to be fully credible.

(A) Less than 8300

(B) At least 8300, but less than 8400

(C) At least 8400, but less than 8500

(D) At least 8500, but less than 8600

(E) At least 8600

Information on Questions 13 and 14 is as follows. You are given the following information on cumulative incurred losses through development years shown.

	Cumulative Incurred Losses				Paid-to-Date
Accident	Development Year				at Dec 31, AY4
Year	0	1	2	3	
AY1	2325	3749	4577	4701	4701
AY2	2657	4438	5529		4500
AY3	2913	4995			3500
AY4	3163				2500

13. Using an average factor model, calculate the estimated total loss reserve as of Dec. 31, AY4.

- (A) Less than 7500
- (B) At least 7500 but less than 7800
- (C) At least 7800 but less than 8100
- (D) At least 8100 but less than 8400
- (E) At least 8400

14. As of Dec. 31, AY4, calculate

Estimated reserve for AY2 based on an average factor model)
− (Estimated reserve for AY2 based on a mean factor model)

- (A) Less than −300
- (B) At least −300 but less than −100
- (C) At least −100 but less than 100
- (D) At least 100 but less than 300
- (E) At least 300

15. For a one-period binomial model for the price of a stock with price 100 at time 0, you are given:

(i) The stock pays no dividends.

(ii) The stock price is either 110 or 95 at the end of the year.

(iii) The risk free force of interest is 5%.

Calculate the price at time 0 of a one-year call option with strike price 100.

- (A) Less than 6.00
- (B) At least 6.00 but less than 6.25
- (C) At least 6.25 but less than 6.50
- (D) At least 6.50 but less than 6.75
- (E) At least 6.75

16. Using the following information, determine the incurred losses for the 2017 accident year as reported at Dec. 31, 2018.

Occurrence #1: Occurrence date Feb. 1/16, Report date Apr. 1/16

Loss History:

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
Apr. 1/16	1000	1000	2000
Dec. 31/16	1500	1000	2500
Dec. 31/17	1500	1000	2500
Mar. 1/18	3000	0	3000

Occurrence #2: Occurrence date May 1/17, Report date July 1/17

Loss History:

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
July 1/17	1000	2000	3000
Dec. 31/17	3000	1000	4000
Dec. 31/18	5000	0	5000

Occurrence #3: Occurrence date Nov. 1/17, Report date Feb. 1/18

Loss History

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
Mar. 1/18	0	8000	8000
Dec. 31/18	5000	5000	10,000

(A) 0 (B) 3,000 (C) 5,000 (D) 8,000 (E) 15,000

17. You are given the following calendar year earned premium.

Year	CY2	CY3	CY4
Earned Premium	4200	4700	5000

You are also given the following rate changes

Date	April 1, CY1	September 1, CY2	July 1, CY3
Average Rate Change	+12 %	+6 %	+10 %

Determine the approximate earned premium at current (end of CY4) rates for CY3.

(A) Less than 5000

(B) At least 5000 but less than 5100

(C) At least 5100 but less than 5200

(D) At least 5200 but less than 5300

(E) At least 5300

**** END OF EXAMINATION ****

Solutions to Practice Exam 1

1.	C	
2.	D	
3.	C	
4.	E	
5.	D	
6.	B	
7.	A	
8.	C	
9.	B	
10.	C	
11.	A	
12.	E	
13.	D	
14.	C	
15.	C	
16.	E	
17.	C	

1. With coinsurance factor α, deductible d, policy limit $\alpha(u-d)$, the amount paid per loss is (we are assuming in inflation rate of $r=0$)$Y = \begin{cases} 0 & X \le d \\ \alpha(X-d) & d < X \le u. \\ \alpha(u-d) & X > u \end{cases}$

   In this problem, the coinsurance factor is $\alpha = .8$, the deductible is $d = 5{,}000$, and the policy limit is $.8(u-5{,}000) = 500{,}000$, so that the maximum covered loss is $u = 630{,}000$.

   The amount paid per loss becomes $Y = \begin{cases} 0 & X \le 5{,}000 \\ 0.80(X-5{,}000) & 5{,}000 < X \le 630{,}000 \\ 500{,}000 & X > 630{,}000 \end{cases}$

   Answer C

2. When a payout occurs, it is 1, 2 or 3 with probability $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}=\frac{7}{8}$. The number of payouts that are 1,2 or 3 follows a Poisson process with an hourly rate of $5\times\frac{7}{8}=\frac{35}{8}$.
   The expected number of payouts that are 1, 2 or 3 in 20 minutes, say N, has a Poisson distribution with mean $\frac{35}{8}\times\frac{20}{60}=\frac{35}{24}$. The probability that there are no payouts of 1, 2, or 3 in a given 20 minute period is the probability that $N=0$, which is $e^{-35/24}=.233$.

   Answer D

3. The minimum claim amount is 5 if a claim occurs. S must be 0 or a multiple of 5 .
   The stop-loss insurance with deductible 6 pays $(S-6)_+ = S-(S\wedge 6)$,
   where $S\wedge 6 = \begin{cases} 0 & S=0 \\ 5 & S=5 \\ 6 & S\ge 10 \end{cases}$

   $E[S]=E[N]\times E[X]=(3)[(5)(.6)+(10)(.4)]=21$
   $E[S\wedge 6]=5\times P(S=5)+6[1-P(S=0{,}5)]$.
   $P(S=0)=P(N=0)=e^{-3}$ and
   $P(S=5)=P(N=1)\times P(X=5)=e^{-3}\times 3\times(.6)=1.8e^{-3}$.
   $E[S\wedge 6]=5\left(1.8e^{-3}\right)+6[1-P(S=0{,}5)]=6\left[1-2.8e^{-3}\right]=5.61$.
   Then $E\left[(S-6)_+\right]=E[S]-E[S\wedge 6]=21-5.61=15.39$.

   Answer C

4. This problem involves a compound distribution.
   The frequency (number of prizes) is N and the severity (prize amount) is X.
   The aggregate prize amount is
   $S=X_1+X_2+\times s+X_N$, with mean $E[S]=E[N]\times E[X]=(1.2)(170)=204$ and variance $Var[S]=E[N]\times Var[X]+Var[N]\times(E[X])^2$

   In this case, $Var[N]=E\left[N^2\right]-(E[N])^2=1.6-(1.2)^2=.16$, and
   $Var[X]=E\left[X^2\right]-(E[X])^2=107{,}000-(170)^2=78{,}100$

   Then, $Var[S]=(1.2)(78{,}100)+(.16)(170)^2=98{,}344$.
   The budget is $E[S]+1.25\sqrt{Var[S]}=204+1.25\sqrt{98{,}344}=596$.

   Answer E

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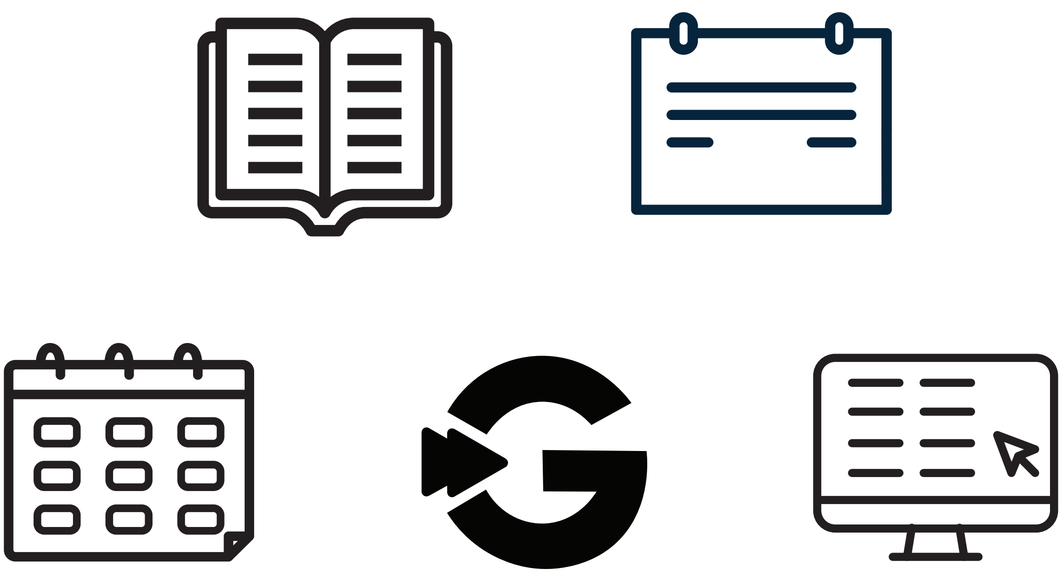

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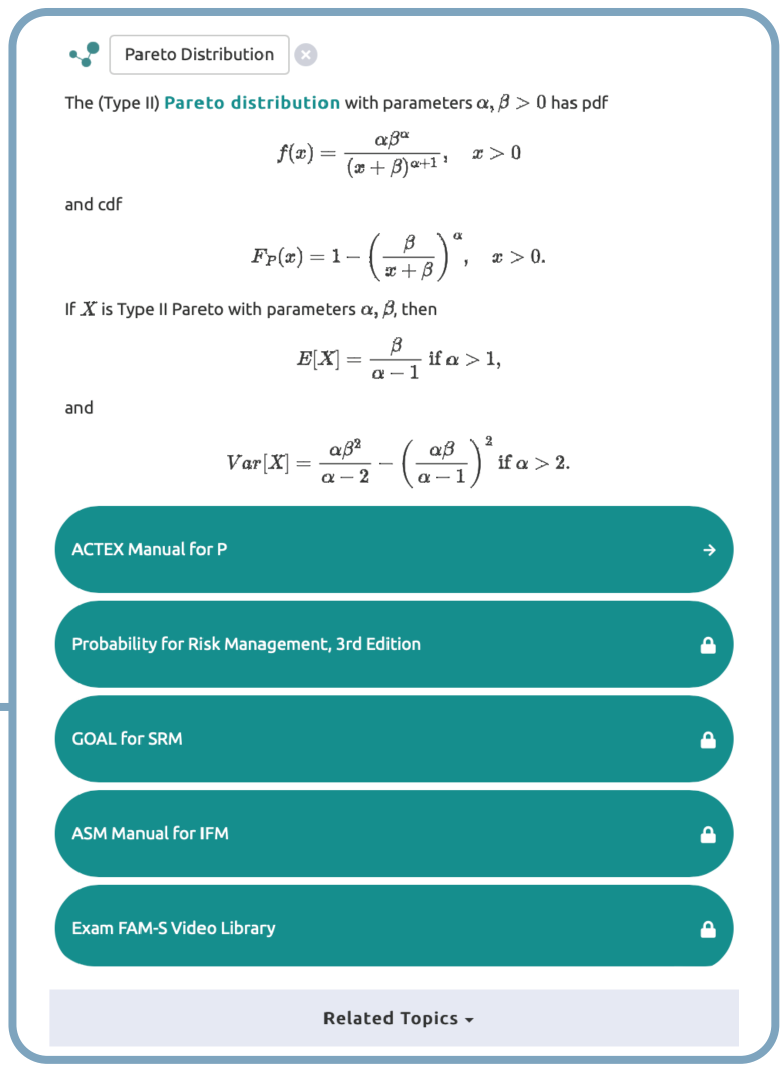

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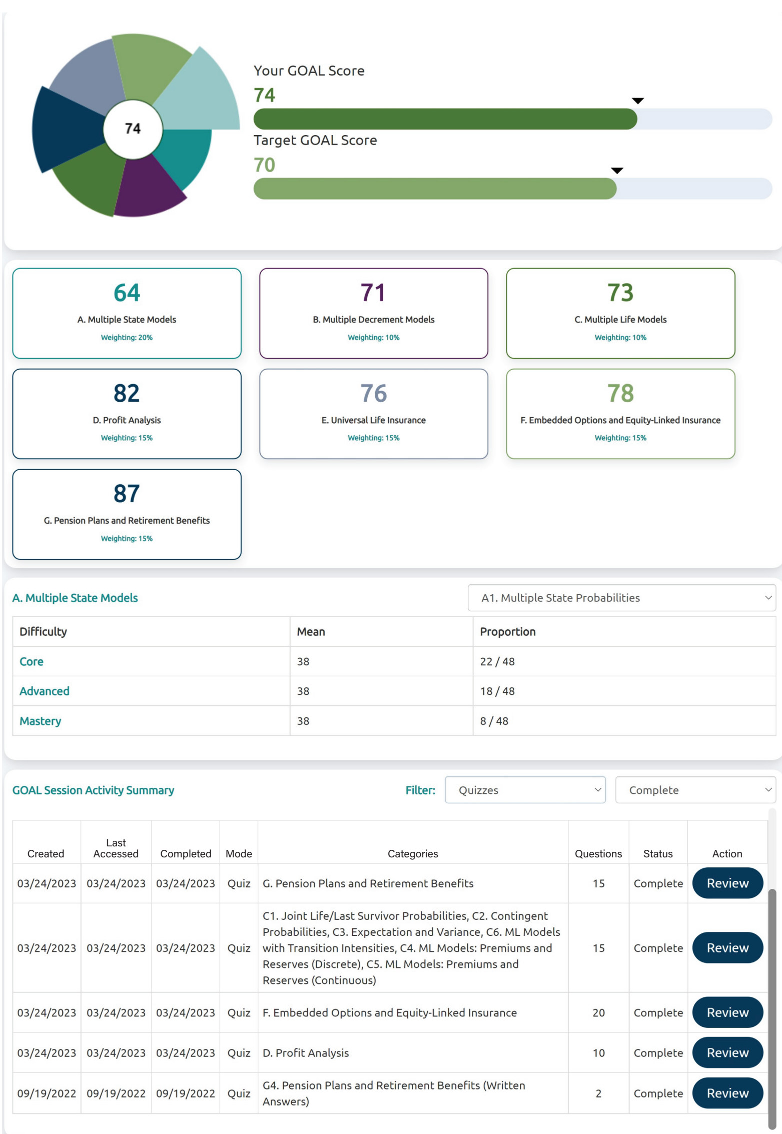

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Preface

Thank you for choosing ACTEX.

In 2022, the SoA launched Exam FAM-L (Fundamentals of Actuarial Mathematics – Long-Term). To help you prepare for this exam, ACTEX developed this brand new study manual, written by Professor Johnny Li and Andrew Ng who have deep knowledge in the exam topics.

Distinguishing features of this study manual include:

- We use graphics extensively. Graphical illustrations are probably the most effective way to explain formulas involved in Exam FAM-L. The extensive use of graphics can also help you remember various concepts and equations.
- A sleek layout is used. The font size and spacing are chosen to let you feel more comfortable in reading. Important equations are displayed in eye-catching boxes.
- Rather than splitting the manual into tiny units, each of which tells you a couple of formulas only, we have carefully grouped the exam topics into 8 chapters and 2 appendices. Such a grouping allows you to more easily identify the linkages between different concepts, which are essential for your success as multiple learning outcomes can be examined in one single exam question.
- Instead of giving you a long list of formulas, we point out which formulas are the most important. Having read this study manual, you will be able to identify the formulas you must remember and the formulas that are just variants of the key ones.
- We do not want to overwhelm you with verbose explanations. Whenever possible, concepts and techniques are demonstrated with examples and integrated into the practice problems.

You should first study all chapters of the study manual in order. Immediately after reading a chapter, do all practice problems we provide for that chapter. Make sure that you understand every single practice problem. Finally, work on the mock exams.

Before you begin your study, please download and read the exam syllabus from the SoA's website:

https://www.soa.org/education/exam-req/edu-exam-fam/

You should also check the exam home page periodically for updates, corrections or notices.

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Enjoy your study!

Chapter 2

Life Tables

Objectives

1. To apply life tables
2. To understand two assumptions for fractional ages: uniform distribution of death and constant force of mortality
3. To calculate moments for future lifetime random variables
4. To understand and model the effect of selection

Actuaries use spreadsheets extensively in practice. It would be very helpful if we could express survival distributions in a tabular form. Such tables, which are known as life tables, are the focus of this chapter.

2.1 Life Table Functions

Below is an excerpt of a (hypothetical) life table. In what follows, we are going to define the functions l_x and d_x, and explain how they are applied.

x	l_x	d_x
0	1000	16
1	984	7
2	977	12
3	965	75
4	890	144

In this hypothetical life table, the value of l_0 is 1,000. This starting value is called the radix of the life table. For $x = 1, 2, \ldots,$ the function l_x stands for the expected number of persons who can survive to age x. Given an assumed value of l_0, we can express any survival function $S_0(x)$ in a tabular form by using the relation

$$l_x = l_0 S_0(x).$$

In the other way around, given the life table function l_x, we can easily obtain values of $S_0(x)$ for integral values of x using the relation

$$S_0(x) = \frac{l_x}{l_0}.$$

Furthermore, we have

$$ {}_tp_x = S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \frac{l_{x+t}/l_0}{l_x/l_0} = \frac{l_{x+t}}{l_x}, $$

which means that we can calculate ${}_tp_x$ for all integral values of t and x from the life table function l_x.

The difference $l_x - l_{x+t}$ is the expected number of deaths over the age interval of $[x, x+t)$. We denote this by ${}_td_x$. It immediately follows that ${}_td_x = l_x - l_{x+t}$.

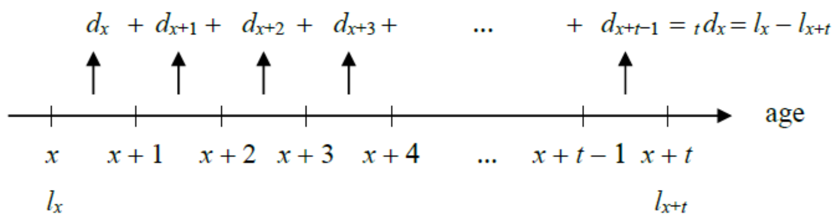
We can then calculate ${}_tq_x$ and ${}_{m|n}q_x$ by the following two relations:

$$ {}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x}, \quad {}_{m|n}q_x = \frac{{}_nd_{x+m}}{l_x} = \frac{l_{x+m} - l_{x+m+n}}{l_x}. $$

When $t = 1$, we can omit the subscript t and write ${}_1d_x$ as d_x. By definition, we have

$$ {}_td_x = d_x + d_{x+1} + \cdots + d_{x+t-1}. $$

Graphically,



Also, when $t = 1$, we have the following relations:

$$ d_x = l_x - l_{x+1}, \; p_x = \frac{l_{x+1}}{l_x}, \text{ and } q_x = \frac{d_x}{l_x}. $$

Summing up, with the life table functions l_x and d_x, we can recover survival probabilities ${}_tp_x$ and death probabilities ${}_tq_x$ for all integral values of t and x easily.

Life Table Functions

(2.1) $$ {}_tp_x = \frac{l_{x+t}}{l_x} $$

(2.2) $$ {}_td_x = l_x - l_{x+t} = d_x + d_{x+1} + \cdots + d_{x+t-1} $$

(2.3) $$ {}_tq_x = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x} = 1 - \frac{l_{x+t}}{l_x} $$

Some FAM exam and LTAM (the predecessor of FAM) questions are based on the Standard Ultimate Life Table, and some MLC (the predecessor of LTAM) exam questions are based on the Illustrative Life Table. The Standard Ultimate Life Table can be found at the SOA's website:

https://www.soa.org/4a481b/globalassets/assets/files/edu/2022/2022-10-exam-fam-l-tables.pdf,

and the Illustrative Life Table is provided in Appendix 2 of this study manual. The two tables have very similar formats. They contain a lot of information. For now, you only need to know and use the first three columns: x, l_x, and q_x (Standard Ultimate Life Table) and $1000q_x$ (Illustrative Life Table). For example, to obtain q_{43}, simply use the column labeled q_x. You should obtain $q_{43} = 0.000656$ (from the Standard Ultimate Life Table). It is also possible, but more tedious, to calculate q_{43} using the column labeled l_x; we have $q_{43} = 1 - l_{44}/l_{43} = 1 - 99104.3/99169.4 = 0.000656452$.

To get values of ${}_tp_x$ and ${}_tq_x$ for $t > 1$, you should always use the column labeled l_x. For example, we have ${}_5p_{61} = l_{66}/l_{61} = 94020.3/96305.8 = 0.976268$ and ${}_5q_{61} = 1 - {}_5p_{61} = 1 - 0.976268 = 0.023732$ (from the Standard Ultimate Life Table). Here, you should not base your calculations on the column labeled q_x, partly because that would be a lot more tedious, and partly because that may lead to a huge rounding error.

Example 2.1. You are given the following excerpt of a life table:

x	l_x	d_x
20	96178.01	99.0569
21	96078.95	102.0149
22	95976.93	105.2582
23	95871.68	108.8135
24	95762.86	112.7102
25	95650.15	116.9802

Calculate the following:

(a) ${}_5p_{20}$

(b) q_{24}

(c) ${}_{4|1}q_{20}$

Solution:

(a) $${}_5p_{20} = \frac{l_{25}}{l_{20}} = \frac{95650.15}{96178.01} = 0.994512.$$

(b) $$q_{24} = \frac{d_{24}}{l_{24}} = \frac{112.7102}{95762.86} = 0.001177.$$

(c) $${}_{4|1}q_{20} = \frac{{}_1d_{24}}{l_{20}} = \frac{112.7102}{96178.01} = 0.001172.$$

□

Example 2.2.

You are given:

(i) $S_0(x) = 1 - \dfrac{x}{100}, \quad 0 \le x \le 100$

(ii) $l_0 = 100$

(a) Find an expression for l_x for $0 \le x \le 100$.

(b) Calculate q_2.

(c) Calculate ${}_3q_2$.

Solution:

(a) $l_x = l_0 S_0(x) = 100 - x$.

(b) $q_2 = \dfrac{l_2 - l_3}{l_2} = \dfrac{98 - 97}{98} = \dfrac{1}{98}$.

(c) ${}_3q_2 = \dfrac{l_2 - l_5}{l_2} = \dfrac{98 - 95}{98} = \dfrac{3}{98}$.

□

In Exam FAM, you may need to deal with a mixture of two populations. As illustrated in the following example, the calculation is a lot more tedious when two populations are involved.

Example 2.3.

For a certain population of 20 year olds, you are given:

(i) 2/3 of the population are nonsmokers, and 1/3 of the population are smokers.

(ii) The future lifetime of a nonsmoker is uniformly distributed over [0, 80).

(iii) The future lifetime of a smoker is uniformly distributed over [0, 50).

Calculate ${}_5p_{40}$ for a life randomly selected from those surviving to age 40.

Solution: The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 20.

- Suppose that there are l_{20} persons in the entire population initially. At time 0 (i.e., at age 20), there are $\frac{2}{3}\,l_{20}$ nonsmokers and $\frac{1}{3}\,l_{20}$ smokers.
- For nonsmokers, the proportion of individuals who can survive to age 40 is $1-20/80 = 3/4$. For smokers, the proportion of individuals who can survive to age 40 is $1-20/50 = 3/5$. At age 40, there are $\frac{3}{4}\frac{2}{3}\,l_{20} = 0.5l_{20}$ nonsmokers and $\frac{3}{5}\frac{1}{3}\,l_{20} = 0.2l_{20}$ smokers. Hence, among those who can survive to age 40, 5/7 are nonsmokers and 2/7 are smokers.

Second, we need to calculate the probabilities of surviving from age 40 to age 45 for both smokers and nonsmokers.

- For a nonsmoker at age 40, the remaining lifetime is uniformly distributed over [0, 60). This means that the probability for a nonsmoker to survive from age 40 to age 45 is $1 - 5/60 = 11/12$.
- For a smoker at age 40, the remaining lifetime is uniformly distributed over [0, 30). This means that the probability for a smoker to survive from age 40 to age 45 is $1 - 5/30 = 5/6$.

Finally, for the whole population, we have

$$ {}_5p_{40} = \frac{5}{7} \times \frac{11}{12} + \frac{2}{7} \times \frac{5}{6} = \frac{25}{28}. $$

□

2.2 Fractional Age Assumptions

We have demonstrated that given a life table, we can calculate values of ${}_tp_x$ and ${}_tq_x$ when both t and x are integers. But what if t and/or x are not integers? In this case, we need to make an assumption about how the survival function behaves between two integral ages. We call such an assumption a fractional age assumption.

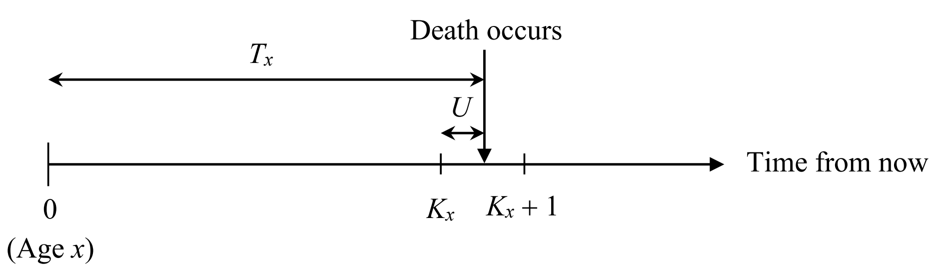
In Exam FAM, you are required to know two fractional age assumptions:

1. Uniform distribution of death
2. Constant force of mortality

We go through these assumptions one by one.

Assumption 1: Uniform Distribution of Death

The Uniform Distribution of Death (UDD) assumption is extensively used in the Exam FAM syllabus. The idea behind this assumption is that we use a bridge, denoted by U, to connect the (continuous) future lifetime random variable T_x and the (discrete) curtate future lifetime random variable K_x. The idea is illustrated diagrammatically as follows:



It is assumed that U follows a uniform distribution over the interval [0, 1], and that U and K_x are independent. Then, for $0 \le r < 1$ and an integral value of x, we have

$$\begin{aligned} {}_rq_x &= \Pr(T_x \le r) \\ &= \Pr(U < r \cap K_x = 0) \\ &= \Pr(U < r)\Pr(K_x = 0) \\ &= rq_x. \end{aligned}$$

The second last step follows from the assumption that U and K_x are independent, while the last step follows from the fact that U follows a uniform distribution over [0, 1].

Key Equation for the UDD Assumption

(2.4) $${}_rq_x = rq_x, \quad \text{for } 0 \le r < 1$$

This means that under UDD, we have, for example, ${}_{0.4}q_{50} = 0.4q_{50}$. The value of q_{50} can be obtained straightforwardly from the life table. To calculate ${}_rp_x$, for $0 \le r < 1$, we use ${}_rp_x = 1 - {}_rq_x = 1 - rq_x$. For example, we have ${}_{0.1}p_{20} = 1 - 0.1q_{20}$.

Equation (2.4) is equivalent to a linear interpolation between l_x and l_{x+1}, that is,

$$l_{x+r} = (1 - r)l_x + rl_{x+1}.$$

Proof:

$$\begin{aligned} {}_rp_x &= 1 - rq_x = (1 - r) + rp_x \\ \frac{l_{x+r}}{l_x} &= (1 - r) + r\frac{l_{x+1}}{l_x} \\ l_{x+r} &= (1 - r)l_x + rl_{x+1} \end{aligned}$$

□

You will find this equation – the interpolation between l_x and l_{x+1} – very useful if you are given a table of l_x (instead of q_x).

Application of the UDD Assumption to l_x

(2.5) $$l_{x+r} = (1 - r)l_x + rl_{x+1}, \quad \text{for } 0 \le r < 1$$

What if the subscript on the left-hand-side of ${}_rq_x$ is greater than 1? In this case, we should first use equation (1.6) from Chapter 1 to break down the probability into smaller portions. As an example, we can calculate ${}_{2.5}p_{30}$ as follows:

$${}_{2.5}p_{30} = {}_2p_{30} \times {}_{0.5}p_{32} = {}_2p_{30} \times (1 - 0.5q_{32}).$$

The value of ${}_2p_{30}$ and q_{32} can be obtained from the life table straightforwardly.

What if the subscript on the right-hand-side is not an integer? In this case, we should make use of a special trick, which we now demonstrate. Let us consider ${}_{0.1}p_{5.7}$ (both subscripts are not integers). The trick is that we multiply this probability with ${}_{0.7}p_5$, that is,

$${}_{0.7}p_5 \times {}_{0.1}p_{5.7} = {}_{0.8}p_5.$$

This gives ${}_{0.1}p_{5.7} = \dfrac{{}_{0.8}p_5}{{}_{0.7}p_5} = \dfrac{1-0.8q_5}{1-0.7q_5}$. The value of q_5 can be obtained from the life table.

To further illustrate this trick, let us consider ${}_{3.5}p_{4.6}$: This probability can be evaluated from the following equation:

$$ {}_{0.6}p_4 \times {}_{3.5}p_{4.6} = {}_{4.1}p_4. $$

Then, we have ${}_{3.5}p_{4.6} = \dfrac{{}_{4.1}p_4}{{}_{0.6}p_4} = \dfrac{{}_4p_4\ {}_{0.1}p_8}{{}_{0.6}p_4} = \dfrac{{}_4p_4(1-0.1q_8)}{1-0.6q_4}$, and finally ${}_{3.5}q_{4.6} = 1 - \dfrac{{}_4p_4(1-0.1q_8)}{1-0.6q_4}$.

The values of ${}_4p_4$, q_8 and q_4 can be obtained from the life table.

Let us study the following example.

Example 2.4. You are given the following excerpt of a life table:

x	l_x	d_x
60	100000	300
61	99700	400
62	99300	500
63	98800	600
64	98200	700
65	97500	800

Assuming uniform distribution of deaths between integral ages, calculate the following:

(a) ${}_{0.26}p_{61}$

(b) ${}_{2.2}q_{60}$

(c) ${}_{0.3}q_{62.8}$

Solution:

(a) ${}_{0.26}p_{61} = 1 - {}_{0.26}q_{61} = 1 - 0.26 \times 400/99700 = 0.998957.$

Alternatively, we can calculate the answer by using a linear interpolation between l_{61} and l_{62} as follows:

$$ l_{61.26} = (1-0.26)l_{61} + 0.26 l_{62} = 0.74 \times 99700 + 0.26 \times 99300 = 99596. $$

It follows that ${}_{0.26}p_{61} = l_{61.26}/l_{61} = 99596/99700 = 0.998957.$

(b) ${}_{2.2}q_{60} = 1 - {}_{2.2}p_{60} = 1 - {}_2p_{60} \times {}_{0.2}p_{62} = 1 - {}_2p_{60} \times (1 - 0.2q_{62})$

$$ = 1 - \frac{l_{62}}{l_{60}}\left(1 - 0.2 \times \frac{d_{62}}{l_{62}}\right) = 1 - \frac{99300}{100000}\left(1 - 0.2 \times \frac{500}{99300}\right) = 0.008. $$

Alternatively, we can calculate the answer by using a linear interpolation between l_{62} and l_{63} as follows:

$$ l_{62.2} = (1-0.2)l_{62} + 0.2 l_{63} = 0.8 \times 99300 + 0.2 \times 98800 = 99200. $$

It follows that ${}_{2.2}q_{60} = 1 - l_{62.2}/l_{60} = 1 - 99200/100000 = 0.008.$

(c) Here, both subscripts are non-integers, so we need to use the trick. First, we compute ${}_{0.3}p_{62.8}$ from the following equation:

$$ {}_{0.8}p_{62} \times {}_{0.3}p_{62.8} = {}_{1.1}p_{62}. $$

Trick

Rearranging the equation above, we have

$$ {}_{0.3}p_{62.8} = \frac{{}_{1.1}p_{62}}{{}_{0.8}p_{62}} = \frac{p_{62}\ {}_{0.1}p_{63}}{{}_{0.8}p_{62}} = \frac{p_{62}\,(1-0.1q_{63})}{1-0.8q_{62}} = \frac{\dfrac{98800}{99300}\left(1-0.1\times\dfrac{600}{98800}\right)}{1-0.8\times\dfrac{500}{99300}} $$

$$ = 0.998382. $$

Hence, ${}_{0.3}q_{62.8} = 1 - 0.998382 = 0.001618$.

Alternatively, we can calculate the answer by using a linear interpolation between l_{62} and l_{63} and another interpolation between l_{63} and l_{64}:

First,

$$ l_{62.8} = (1-0.8)l_{62} + 0.8l_{63} = 0.2 \times 99300 + 0.8 \times 98800 = 98900. $$

Second,

$$ l_{63.1} = (1-0.1)l_{63} + 0.1l_{64} = 0.9 \times 98800 + 0.1 \times 98200 = 98740. $$

Finally,

$$ {}_{0.3}q_{62.8} = 1 - {}_{0.3}p_{62.8} = 1 - l_{63.1}/l_{62.8} = 1 - 98740/98900 = 0.001618. $$

□

Sometimes, you may be asked to calculate the density function of T_x and the force of mortality from a life table. Under UDD, we have the following equation for calculating the density function:

$$ f_x(r) = q_x, \quad 0 \le r < 1. $$

Proof: $f_x(r) = \frac{\mathrm{d}}{\mathrm{d}r}F_x(r) = \frac{\mathrm{d}}{\mathrm{d}r}\Pr(T_x \le r) = \frac{\mathrm{d}}{\mathrm{d}r}{}_rq_x = \frac{\mathrm{d}}{\mathrm{d}r}(rq_x) = q_x.$ □

Under UDD, we have the following equation for calculating the force of mortality:

$$ \mu_{x+r} = \frac{q_x}{1-rq_x}, \quad 0 \le r < 1. $$

Proof: In general, $f_x(r) = {}_rp_x\mu_{x+r}$. Under UDD, we have $f_x(r) = q_x$ and ${}_rp_x = 1 - rq_x$. The result follows. □

Let us take a look at the following example.

Example 2.5. **[Course 3 Spring 2000 #12]**

For a certain mortality table, you are given:

(i) $\mu_{80.5} = 0.0202$

(ii) $\mu_{81.5} = 0.0408$

(iii) $\mu_{82.5} = 0.0619$

(iv) Deaths are uniformly distributed between integral ages.

Calculate the probability that a person age 80.5 will die within two years.

(A) 0.0782 (B) 0.0785 (C) 0.0790 (D) 0.0796 (E) 0.0800

Solution: The probability that a person age 80.5 will die within two years is ${}_2q_{80.5}$. We have

$$ {}_{0.5}p_{80} \times {}_2p_{80.5} = {}_{2.5}p_{80}. $$

This gives

$$ {}_2p_{80.5} = \frac{{}_2p_{80}\ {}_{0.5}p_{82}}{{}_{0.5}p_{80}} = \frac{p_{80}p_{81}\left(1-0.5q_{82}\right)}{1-0.5q_{80}} = \frac{\left(1-q_{80}\right)\left(1-q_{81}\right)\left(1-0.5q_{82}\right)}{1-0.5q_{80}}. $$

We then need to find q_{80}, q_{81} and q_{82} from the information given in the question. Using $\mu_{80.5}$, we have $\mu_{80.5} = \dfrac{q_{80}}{1-0.5q_{80}} \Rightarrow q_{80} = 0.0200$. Similarly, by using $\mu_{81.5}$ and $\mu_{82.5}$, we obtain $q_{81} = 0.0400$ and $q_{82} = 0.0600$.

Substituting q_{80}, q_{81} and q_{82}, we obtain ${}_2p_{80.5} = 0.921794$, and hence ${}_2q_{80.5} = 1 - {}_2p_{80.5} = 0.0782$. Hence, the answer is (A).

□

Assumption 2: Constant Force of Mortality

The idea behind this assumption is that for every age x, we approximate μ_{x+r} for $0 \le r < 1$ by a constant, which we denote by $\tilde{\mu}_x$. This means

$$ \int_0^1 \mu_{x+u}\ \mathrm{d}u = \int_0^1 \tilde{\mu}_x\ \mathrm{d}u = \tilde{\mu}_x, $$

which implies $p_x = e^{-\tilde{\mu}_x}$ and $\tilde{\mu}_x = -\ln\left(p_x\right)$.

We are now ready to develop equations for calculating various death and survival probabilities. First of all, for any integer-valued x, we have

$$ {}_rp_x = \left(p_x\right)^r, \quad 0 \le r < 1. $$

Proof: ${}_rp_x = \exp\left(-\int_0^r \mu_{x+u}\mathrm{d}u\right) = \exp\left(-\int_0^r \tilde{\mu}_x\mathrm{d}u\right) = e^{-\tilde{\mu}_x r} = \left(e^{-\tilde{\mu}_x}\right)^r = \left(p_x\right)^r.$ □

For example, ${}_{0.3}p_{50} = \left(p_{50}\right)^{0.3}$, and ${}_{0.4}q_{62} = 1 - {}_{0.4}p_{62} = 1 - \left(p_{62}\right)^{0.4}$. We can generalize the equation above to obtain the following key formula.

Key Equation for the Constant Force of Mortality Assumption

(2.6) $${}_rp_{x+u} = (p_x)^r, \quad \text{for } 0 \le r < 1 \text{ and } r+u \le 1$$

Proof: ${}_rp_{x+u} = \exp\left(-\int_0^r \mu_{x+u+t}\mathrm{d}t\right) = \exp\left(-\int_0^r \tilde{\mu}_x \mathrm{d}t\right) = e^{-\tilde{\mu}_x r} = (p_x)^r$.

[The second step follows from the fact that given $0 \le r < 1$, $u+t$ is always less than or equal to 1 when $0 \le t \le r$.] □

Notice that the key equation for the constant force of mortality assumption is based on p, while that for the UDD assumption is based on q.

This key equation means that, for example, ${}_{0.2}p_{30.3} = (p_{30})^{0.2}$. Note that the subscript u on the right-hand-side does not appear in the result, provided that the condition $r+u \le 1$ is satisfied. But what if $r+u > 1$? The answer is very simple: Split the probability! To illustrate, let us consider ${}_{0.8}p_{30.3}$. (Here, $r+u = 0.8+0.3 = 1.1 > 1$.) By using equation (1.6) from Chapter 1, we can split ${}_{0.8}p_{30.3}$ into two parts as follows:

$${}_{0.8}p_{30.3} = {}_{0.7}p_{30.3} \times {}_{0.1}p_{31}.$$

We intentionally consider a duration of 0.7 years for the first part, because $0.3+0.7 = 1$, which means we can apply the key equation ${}_rp_{x+u} = (p_x)^r$ to it. As a result, we have

$${}_{0.8}p_{30.3} = (p_{30})^{0.7} \times (p_{31})^{0.1}.$$

The values of p_{30} and p_{31} can be obtained from the life table straightforwardly.

To further illustrate, let us consider ${}_{5.6}p_{40.8}$. We can split it as follows:

$${}_{5.6}p_{40.8} = {}_{0.2}p_{40.8} \times {}_{5.4}p_{41} = {}_{0.2}p_{40.8} \times {}_5p_{41} \times {}_{0.4}p_{46} = (p_{40})^{0.2} \times {}_5p_{41} \times (p_{46})^{0.4}.$$

The values of p_{40}, ${}_5p_{41}$ and p_{46} can be obtained from the life table straightforwardly.

Interestingly, equation (2.6) implies that for $0 \le r < 1$, the value of $\ln(l_{x+r})$ can be obtained by a linear interpolation between the values of $\ln(l_x)$ and $\ln(l_{x+1})$.

Proof: Setting $u = 0$ in equation (2.6), we have

$$\begin{aligned} {}_rp_x &= (p_x)^r \\ \frac{l_{x+r}}{l_x} &= \left(\frac{l_{x+1}}{l_x}\right)^r \\ \ln(l_{x+r}) - \ln(l_x) &= r\ln(l_{x+1}) - r\ln(l_x) \\ \ln(l_{x+r}) &= (1-r)\ln(l_x) + r\ln(l_{x+1}) \end{aligned}$$

□

You will find this equation – the interpolation between $\ln(l_x)$ and $\ln(l_{x+1})$ – useful when you are given a table of l_x.

Application of the Constant Force of Mortality Assumption to l_x

$$\ln(l_{x+r}) = (1-r)\ln(l_x) + r\ln(l_{x+1}), \quad \text{for } 0 \le r < 1$$

Example 2.6.

Assuming constant force of mortality between integral ages, repeat Example 2.4.

Solution:

(a) ${}_{0.26}p_{61} = (p_{61})^{0.26} = (99300/99700)^{0.26} = 0.998955.$

Alternatively, we can calculate the answer by interpolating between $\ln(l_{61})$ and $\ln(l_{62})$ as follows: $\ln(l_{61.26}) = (1-0.26)\ln(l_{61}) + 0.26\ln(l_{62})$, which gives $l_{61.26} = 99595.84526$. Hence, ${}_{0.26}p_{61} = l_{61.26}/l_{61} = 99595.84526/99700 = 0.998955$.

(b) ${}_{2.2}q_{60} = 1 - {}_{2.2}p_{60} = 1 - {}_{2}p_{60} \times {}_{0.2}p_{62} = 1 - {}_{2}p_{60} \times (p_{62})^{0.2}$

$$= 1 - \frac{l_{62}}{l_{60}}\left(\frac{l_{63}}{l_{62}}\right)^{0.2} = 1 - \frac{99300}{100000}\left(\frac{98800}{99300}\right)^{0.2} = 0.008002.$$

Alternatively, we can calculate the answer by interpolating between $\ln(l_{62})$ and $\ln(l_{63})$ as follows: $\ln(l_{62.2}) = (1-0.2)\ln(l_{62}) + 0.2\ln(l_{63})$, which gives $l_{62.2} = 99199.79798$. Hence, ${}_{2.2}q_{60} = 1 - l_{62.2}/l_{60} = 0.008002$.

(c) First, we consider ${}_{0.3}p_{62.8}$:

$${}_{0.3}p_{62.8} = {}_{0.2}p_{62.8} \times {}_{0.1}p_{63} = (p_{62})^{0.2}(p_{63})^{0.1}.$$

Hence,

$${}_{0.3}q_{62.8} = 1 - (p_{62})^{0.2}(p_{63})^{0.1} = 1 - \left(\frac{l_{63}}{l_{62}}\right)^{0.2}\left(\frac{l_{64}}{l_{63}}\right)^{0.1}$$

$$= 1 - \left(\frac{98800}{99300}\right)^{0.2}\left(\frac{98200}{98800}\right)^{0.1} = 0.001617.$$

Alternatively, we can calculate the answer by an interpolation between $\ln(l_{62})$ and $\ln(l_{63})$ and another interpolation between $\ln(l_{63})$ and $\ln(l_{64})$.

First, $\ln(l_{62.8}) = (1-0.8)\ln(l_{62}) + 0.8\ln(l_{63})$, which gives $l_{62.8} = 98899.79818$.

Second, $\ln(l_{63.1}) = (1-0.1)\ln(l_{63}) + 0.1\ln(l_{64})$, which gives $l_{63.1} = 98739.8354$.

Finally, ${}_{0.3}q_{62.8} = 1 - l_{63.1}/l_{62.8} = 0.001617$.

□

Example 2.7.

You are given the following life table:

x	l_x	d_x
90	1000	50
91	950	50
92	900	60
93	840	c_1
94	c_2	70
95	700	80

(a) Find the values of c_1 and c_2.

(b) Calculate ${}_{1.4}p_{90}$, assuming uniform distribution of deaths between integer ages.

(c) Repeat (b) by assuming constant force of mortality between integer ages.

Solution:

(a) We have $840 - c_1 = c_2$ and $c_2 - 70 = 700$. This gives $c_2 = 770$ and $c_1 = 70$.

(b) Assuming uniform distribution of deaths between integer ages, we have

$$\begin{aligned}{}_{1.4}p_{90} &= p_{90} \times {}_{0.4}p_{91} \\ &= p_{90}\,(1 - 0.4q_{91}) \\ &= \frac{l_{91}}{l_{90}}\left(1 - 0.4\frac{d_{91}}{l_{91}}\right) \\ &= \frac{950}{1000}\left(1 - 0.4 \times \frac{50}{950}\right) \\ &= 0.93.\end{aligned}$$

Alternatively, you can compute the answer by interpolating between l_{92} and l_{91}:

$${}_{1.4}p_{90} = p_{90} \times {}_{0.4}p_{91} = \frac{l_{91}}{l_{90}}\left(\frac{0.4l_{92} + 0.6l_{91}}{l_{91}}\right) = \frac{0.4 \times 900 + 0.6 \times 950}{1000} = 0.93$$

(c) Assuming constant force of mortality between integer ages, we have

$$\begin{aligned}{}_{1.4}p_{90} &= p_{90} \times_{0.4} p_{91} \\ &= p_{90} \times (p_{91})^{0.4} \\ &= \frac{950}{1000}\left(\frac{900}{950}\right)^{0.4} \\ &= 0.92968.\end{aligned}$$

□

Let us conclude this section with the following table, which summarizes the formulas for the two fractional age assumptions.

	UDD	**Constant force**
${}_r p_x$	$1 - r q_x$	$(p_x)^r$
${}_r q_x$	$r q_x$	$1 - (p_x)^r$
μ_{x+r}	$\dfrac{q_x}{1 - r q_x}$	$-\ln(p_x)$

In the table, x is an integer and $0 \leq r < 1$. The shaded formulas are the key formulas that you must remember for the examination.

2.3 Select-and-Ultimate Tables

Insurance companies typically assess risk before they agree to insure you. They cannot stay in business if they sell life insurance to someone who has just discovered he has only a few months to live. A team of underwriters will usually review information about you before you are sold insurance (although there are special insurance types called "guaranteed issue" which cannot be underwritten). For this reason, a person who has just purchased life insurance has a lower probability of death than a person the same age in the general population. The probability of death for a person who has just been issued life insurance is called a select probability. In this section, we focus on the modeling of select probabilities.

Let us define the following notation.

- $[x]$ indicates the age at selection (i.e., the age at which the underwriting was done).
- $[x]+t$ indicates a person currently age $x+t$ who was selected at age x (i.e., the underwriting was done at age x). This implies that the insurance contract has elapsed for t years.

For example, we have the following select probabilities:

- $q_{[x]}$ is the probability that a life age x now dies before age $x + 1$, given that the life is selected at age x.
- $q_{[x]+t}$ is the probability that a life age $x + t$ now dies before age $x + t + 1$, given that the life was selected at age x.

Due to the effect of underwriting, a select death probability $q_{[x]+t}$ must be no greater than the corresponding ordinary death probability q_{x+t}. However, the effect of underwriting will not last forever. The period after which the effect of underwriting is completely gone is called the select period. Suppose that the select period is n years, we have

$$q_{[x]+t} < q_{x+t}, \quad \text{for } t < n.$$
$$q_{[x]+t} = q_{x+t}, \quad \text{for } t \geq n.$$

The ordinary death probability q_{x+t} is called the ultimate death probability. A life table that contains both select probabilities and ultimate probabilities is called a select-and-ultimate life table. The following is an excerpt of a (hypothetical) select-and-ultimate table with a select period of two years.

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
40	0.04	0.06	0.08	42
41	0.05	0.07	0.09	43
42	0.06	0.08	0.10	44
43	0.07	0.09	0.11	45

It is important to know how to apply such a table. Let us consider a person who is currently age 41 and is just selected. The death probabilities for this person are as follows:

Age 41: $q_{[41]} = 0.05$

Age 42: $q_{[41]+1} = 0.07$

Age 43: $q_{[41]+2} = q_{43} = 0.09$

Age 44: $q_{[41]+3} = q_{44} = 0.10$

Age 45: $q_{[41]+4} = q_{45} = 0.11$

As you see, the select-and-ultimate table is not difficult to use. We progress horizontally until we reach the ultimate death probability, then we progress vertically as when we are using an ordinary life table. To further illustrate, let us consider a person who is currently age 41 and was selected at age 40. The death probabilities for this person are as follows:

Age 41: $q_{[40]+1} = 0.06$

Age 42: $q_{[40]+2} = q_{42} = 0.08$

Age 43: $q_{[40]+3} = q_{43} = 0.09$

Age 44: $q_{[40]+4} = q_{44} = 0.10$

Age 45: $q_{[40]+5} = q_{45} = 0.11$

Even though the two persons we considered are of the same age now, their current death probabilities are different. Because the first individual has the underwriting done more recently, the effect of underwriting on him/her is stronger, which means he/she should have a lower death probability than the second individual.

We may measure the effect of underwriting by the index of selection, which is defined as follows:

$$I(x,k) = 1 - \frac{q_{[x]+k}}{q_{x+k}}.$$

For example, on the basis of the preceding table, $I(41,1) = 1 - q_{[41]+1}/q_{42} = 1 - 0.07/0.08 = 0.125$. If the effect of underwriting is strong, then $q_{[x]+k}$ would be small compared to q_{x+k}, and therefore $I(x,k)$ would be close to one. By contrast, if the effect of underwriting is weak, then $q_{[x]+k}$ would be close to q_{x+k}, and therefore $I(x,k)$ would be close to zero.

Let us first go through the following example, which involves a table of $q_{[x]}$.

Example 2.8. **[Course 3 Fall 2001 #2]**

For a select-and-ultimate mortality table with a 3-year select period:

(i)

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
60	0.09	0.11	0.13	0.15	63
61	0.10	0.12	0.14	0.16	64
62	0.11	0.13	0.15	0.17	65
63	0.12	0.14	0.16	0.18	66
64	0.13	0.15	0.17	0.19	67

(ii) White was a newly selected life on 01/01/2000.

(iii) White's age on 01/01/2001 is 61.

(iv) P is the probability on 01/01/2001 that White will be alive on 01/01/2006.

Calculate P.

(A) $0 \leq P < 0.43$

(B) $0.43 \leq P < 0.45$

(C) $0.45 \leq P < 0.47$

(D) $0.47 \leq P < 0.49$

(E) $0.49 \leq P < 1.00$

Solution: White is now age 61 and was selected at age 60. So the probability that White will be alive 5 years from now can be expressed as $P = {}_5p_{[60]+1}$. We have

$$\begin{aligned} P &= {}_5p_{[60]+1} \\ &= p_{[60]+1} \times p_{[60]+2} \times p_{[60]+3} \times p_{[60]+4} \times p_{[60]+5} \\ &= p_{[60]+1} \times p_{[60]+2} \times p_{63} \times p_{64} \times p_{65} \\ &= (1 - q_{[60]+1})(1 - q_{[60]+2})(1 - q_{63})(1 - q_{64})(1 - q_{65}) \\ &= (1 - 0.11)(1 - 0.13)(1 - 0.15)(1 - 0.16)(1 - 0.17) \\ &= 0.4589. \end{aligned}$$

Hence, the answer is (C).

□

In some exam questions, a select-and-ultimate table may be used to model a real life problem. Take a look at the following example.

Example 2.9. **[MLC Spring 2012 #13]**

Lorie's Lorries rents lavender limousines.

On January 1 of each year they purchase 30 limousines for their existing fleet; of these, 20 are new and 10 are one-year old.

Vehicles are retired according to the following 2-year select-and-ultimate table, where selection is age at purchase:

Limousine age (x)	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
0	0.100	0.167	0.333	2
1	0.100	0.333	0.500	3
2	0.150	0.400	1.000	4
3	0.250	0.750	1.000	5
4	0.500	1.000	1.000	6
5	1.000	1.000	1.000	7

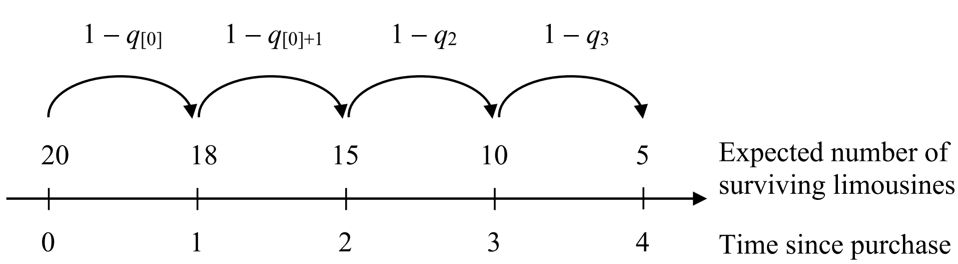
Lorie's Lorries has rented lavender limousines for the past ten years and has always purchased its limousines on the above schedule.

Calculate the expected number of limousines in the Lorie's Lorries fleet immediately after the purchase of this year's limousines.

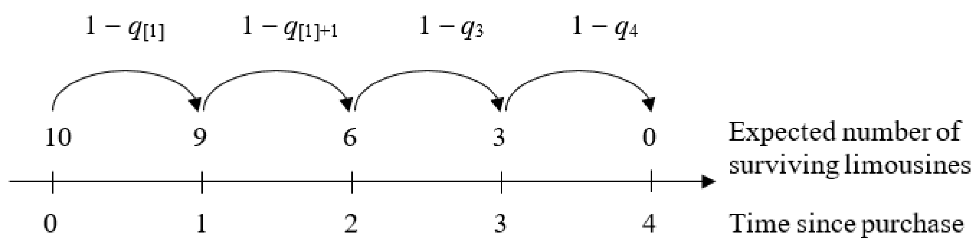
(A) 93 (B) 94 (C) 95 (D) 96 (E) 97

Solution: Let us consider a purchase of 30 limousines in a given year. According to information given, 20 of them are brand new while 10 of them are 1-year-old.

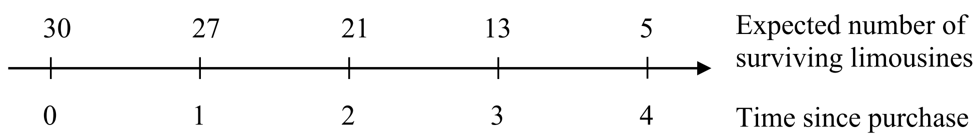
For the 20 brand new limousines, their "age at selection" is 0. As such, the sequence of "death" probabilities applicable to these 20 new limousines are $q_{[0]}$, $q_{[0]+1}$, q_2, q_3, q_4, q_5, ... Note that $q_4 = q_5 = \ldots = 1$, which implies that these limousines can last for at most four years since the time of purchase. For these 20 brand new limousines, the expected number of "survivors" limousines in each future year can be calculated as follows:



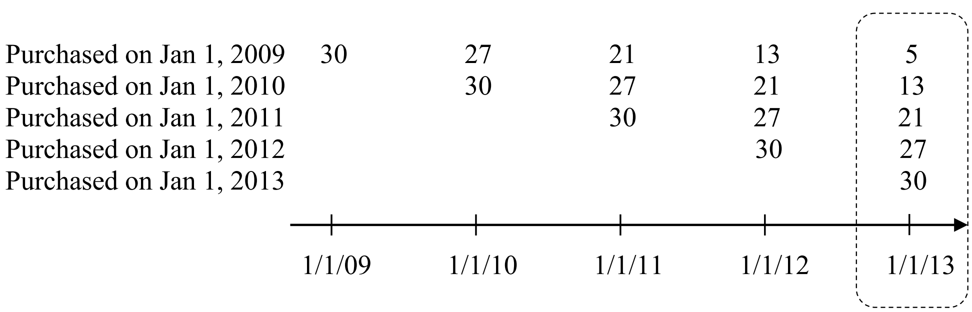
For the 10 1-year-old limousines, their "age at selection" is 1. As such, the sequence of "death" probabilities applicable to these 10 1-year-old limousines are $q_{[1]}$, $q_{[1]+1}$, q_3, q_4, ... Note that $q_4 = q_5 = \ldots = 1$, which implies that these limousines can last for at most three years since the time of purchase. For these 10 1-year-old limousines, the expected number of "surviving" limousines in each future year can be calculated as follows:



Considering the entire purchase of 30 limousines, we have the following:



Suppose that today is January 1, 2013. Since a limousine cannot last for more than four years since the time of purchase, the oldest limousine in Lorie's fleet should be purchased on January 1, 2009. Using the results above, the expected number of limousines on January 1, 2013 can be calculated as follows:



The answer is thus $5 + 13 + 21 + 27 + 30 = 96$, which corresponds to option (D). □

Sometimes, you may be given a select-and-ultimate table that contains the life table function l_x. In this case, you can calculate survival and death probabilities by using the following equations:

$$ {}_sp_{[x]+t} = \frac{l_{[x]+t+s}}{l_{[x]+t}}, \quad {}_sq_{[x]+t} = 1 - \frac{l_{[x]+t+s}}{l_{[x]+t}}. $$

Let us study the following two examples.

Example 2.10. You are given the following select-and-ultimate table with a 2-year select period:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
30	9907	9905	9901	32
31	9903	9900	9897	33
32	9899	9896	9892	34
33	9894	9891	9887	35

Calculate the following:

(a) ${}_2q_{[31]}$

(b) ${}_2p_{[30]+1}$

(c) ${}_{1|2}q_{[31]+1}$

Solution:

(a) ${}_2q_{[31]} = 1 - \dfrac{l_{[31]+2}}{l_{[31]}} = 1 - \dfrac{l_{33}}{l_{[31]}} = 1 - \dfrac{9897}{9903} = 0.000606.$

(b) ${}_2p_{[30]+1} = \dfrac{l_{[30]+1+2}}{l_{[30]+1}} = \dfrac{l_{33}}{l_{[30]+1}} = \dfrac{9897}{9905} = 0.999192.$

(c) ${}_{1|2}q_{[31]+1} = \dfrac{l_{[31]+1+1} - l_{[31]+1+1+2}}{l_{[31]+1}} = \dfrac{l_{33} - l_{35}}{l_{[31]+1}} = \dfrac{9897 - 9887}{9900} = 0.001010.$ □

Exam questions such as the following may involve both $q_{[x]}$ and $l_{[x]}$.

Example 2.11. **[MLC Spring 2012 #1]**

For a 2-year select and ultimate mortality model, you are given:

(i) $q_{[x]+1} = 0.95q_{x+1}$

(ii) $l_{76} = 98{,}153$

(iii) $l_{77} = 96{,}124$

Calculate $l_{[75]+1}$.

(A) 96,150 (B) 96,780 (C) 97,420 (D) 98,050 (E) 98,690

Solution: From (ii) and (iii), we know that $q_{76} = 1 - 96124/98153 = 0.020672$.

From (i), we know that $q_{[75]+1} = 0.95q_{76} = 0.95 \times 0.020672 = 0.019638$.

Since

$$l_{[75]+2} = l_{[75]+1}(1 - q_{[75]+1}),$$

and $l_{[75]+2} = l_{77}$, we have $l_{[75]+1} = 96124/(1 - 0.019638) = 98049.5$. The answer is (D). □

It is also possible to set questions to examine your knowledge on select-and-ultimate tables and fractional age assumptions at the same time. The next example involves a select-and-ultimate table and the UDD assumption.

Example 2.12. [Course 3 Fall 2000 #10]

You are given the following extract from a select-and-ultimate mortality table with a 2-year select period:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
60	80625	79954	78839	62
61	79137	78402	77252	63
62	77575	76770	75578	64

Assume that deaths are uniformly distributed between integral ages.

Calculate ${}_{0.9}q_{[60]+0.6}$.

(A) 0.0102 (B) 0.0103 (C) 0.0104 (D) 0.0105 (E) 0.0106

Solution: We illustrate two methods:

(1) Interpolation

The live age $q_{[60]+0.6}$ is originally selected at age [60]. So, we can use $l_{[60]} = 80625, l_{[60]+1} = 79954, l_{[60]+2} = l_{62} = 78839$ and so on to calculate mortality rate.

$$ {}_{0.9}q_{[60]+0.6} = 1 - \frac{l_{[60]+1.5}}{l_{[60]+0.6}} $$

$$ \begin{aligned} l_{[60]+0.6} &= 0.4l_{[60]} + 0.6l_{[60]+1} \\ &= 0.4 \times 80625 + 0.6 \times 79954 \\ &= 80222.4 \end{aligned} $$

$$ \begin{aligned} l_{[60]+1.5} &= 0.5l_{[60]+1} + 0.5l_{[60]+2} \\ &= 0.5 \times 79954 + 0.6 \times 78839 \\ &= 79396.5 \end{aligned} $$

The death probability is $1 - \dfrac{79396.5}{80222.4} = 0.010295$.

(2) The trick we have introduced to shift the fractional age to integral age

Recall that when UDD is assumed and the subscript on the right-hand-side is not an integer, we will need to use the trick. We first calculate ${}_{0.9}p_{[60]+0.6}$. Using the trick, we have

$$ {}_{0.6}p_{[60]} \times {}_{0.9}p_{[60]+0.6} = {}_{1.5}p_{[60]}. $$

Then, we have

$$
\begin{aligned}
{}_{0.9}p_{[60]+0.6} &= \frac{{}_{1.5}p_{[60]}}{{}_{0.6}p_{[60]}} = \frac{p_{[60]}\ {}_{0.5}p_{[60]+1}}{{}_{0.6}p_{[60]}} \\
&= \frac{p_{[60]}\left(1-0.5q_{[60]+1}\right)}{1-0.6q_{[60]}} \\
&= \frac{\dfrac{l_{[60]+1}}{l_{[60]}}\left[1-0.5\left(1-\dfrac{l_{[60]+2}}{l_{[60]+1}}\right)\right]}{1-0.6\left(1-\dfrac{l_{[60]+1}}{l_{[60]}}\right)} \\
&= \frac{\dfrac{79954}{80625}\left[1-0.5\left(1-\dfrac{78839}{79954}\right)\right]}{1-0.6\left(1-\dfrac{79954}{80625}\right)} \\
&= 0.989705.
\end{aligned}
$$

As a result, ${}_{0.9}q_{[60]+0.6} = 1 - 0.989705 = 0.0103$. Hence, the answer is (B).

□

The following example involves a select-and-ultimate table and the constant force of mortality assumption.

Example 2.13. **[MLC Fall 2012 #2]**

You are given:

(i) An excerpt from a select and ultimate life table with a select period of 3 years.

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}	$x+3$
60	80,000	79,000	77,000	74,000	63
61	78,000	76,000	73,000	70,000	64
62	75,000	72,000	69,000	67,000	65
63	71,000	68,000	66,000	65,000	66

(ii) Deaths follow a constant force of mortality over each year of age.

Calculate $1000\ {}_{2|3}q_{[60]+0.75}$.

(A) 104 (B) 117 (C) 122 (D) 135 (E) 142

Solution: As discussed in Section 2.2, there are two methods for solving such a problem.

Method 1: Interpolation

The probability required is ${}_{2|3}q_{[60]+0.75} = \dfrac{l_{[60]+2.75} - l_{[60]+5.75}}{l_{[60]+0.75}} = \dfrac{l_{[60]+2.75} - l_{65.75}}{l_{[60]+0.75}}$.

Under the constant force of mortality assumption, we have

$$\begin{aligned}\ln l_{[60]+0.75} &= 0.25\ln l_{[60]} + 0.75\ln l_{[60]+1} = 0.25\ln 80000 + 0.75\ln 79000\\ &\Rightarrow l_{[60]+0.75} = \exp(11.28035) = 79248.82\end{aligned}$$

$$\begin{aligned}\ln l_{[60]+2.75} &= 0.25\ln l_{[60]+2} + 0.75\ln l_{63} = 0.25\ln 77000 + 0.75\ln 74000\\ &\Rightarrow l_{[60]+2.75} = \exp(11.22176) = 74738.86\end{aligned}$$

$$\begin{aligned}\ln l_{65.75} &= 0.25\ln l_{65} + 0.75\ln l_{66} = 0.25\ln 67000 + 0.75\ln 65000\\ &\Rightarrow l_{63.75} = \exp(11.08972) = 65494.33\end{aligned}$$

As a result, ${}_{2|3}q_{[60]+0.75} = (74738.86 - 65494.33)/79248.82 = 0.11665$.

<u>Method 2</u>: Working on the survival probabilities

The probability required is

$$\begin{aligned}{}_{2|3}q_{[60]+0.75} &= {}_{2}p_{[60]+0.75} - {}_{5}p_{[60]+0.75}\\ &= {}_{0.25}p_{[60]+0.75} \times p_{[60]+1} \times {}_{0.75}p_{[60]+2} - {}_{0.25}p_{[60]+0.75} \times {}_{4}p_{[60]+1} \times {}_{0.75}p_{[60]+5}\\ &= \left(\frac{79}{80}\right)^{1/4} \times \frac{77}{79} \times \left(\frac{74}{77}\right)^{0.75} - \left(\frac{79}{80}\right)^{1/4} \times \frac{67}{79} \times \left(\frac{65}{67}\right)^{0.75}\\ &= 0.11665\end{aligned}$$

Both methods imply $1000\ {}_{2|3}q_{[60]+0.75} = 116.65$, which corresponds to option (B).

□

2.4 Moments of Future Lifetime Random Variables

In Exam P, you learnt how to calculate the moments of a random variable.

- If W is a discrete random variable, then $\mathrm{E}(W) = \sum_{w} w\Pr(W = w)$.
- If W is a continuous random variable, then $\mathrm{E}(W) = \int_{-\infty}^{\infty} w f_W(w)\mathrm{d}w$, where $f_W(w)$ is the density function for W.
- To calculate variance, we can always use the identity $\mathrm{Var}(W) = \mathrm{E}(W^2) - [\mathrm{E}(W)]^2$.

First, let us focus on the moments of the future lifetime random variable T_x. We call $\mathrm{E}(T_x)$ the complete expectation of life at age x, and denote it by $\mathring{e}_x$. We have

$$\mathring{e}_x = \int_0^{\infty} t f_x(t)\mathrm{d}t = \int_0^{\infty} t\,{}_tp_x\mu_{x+t}\mathrm{d}t.$$

By rewriting the integral as $-\int_0^{\infty} t\mathrm{d}S_x(t)$ and using integration by parts, we have

$$\mathring{e}_x = -[tS_x(t)]_0^{\infty} + \int_0^{\infty} S_x(t)\mathrm{d}t = \int_0^{\infty} {}_tp_x\mathrm{d}t.$$

Note that if there is a limiting age, we replace ∞ with $\omega - x$.

The second moment of T_x can be expressed as

$$\mathrm{E}(T_x^2) = \int_0^{\infty} t^2 f_x(t)\mathrm{d}t.$$

Using integration by parts, we can show that the above formula can be rewritten as

$$\mathrm{E}(T_x^2) = 2\int_0^{\infty} t\ {}_tp_x\mathrm{d}t,$$

which is generally easier to apply. Again, if there is a limiting age, we replace ∞ with $\omega - x$.

In the exam, you may also be asked to calculate $\mathrm{E}(T_x \wedge n) = \mathrm{E}[\min(T_x, n)]$. This expectation is known as the n-year temporary complete expectation of life at age x, and is denoted by $\mathring{e}_{x:\overline{n|}}$. By definition,

$$\mathring{e}_{x:\overline{n|}} = \int_0^{n} t f_x(t)\mathrm{d}t + \int_n^{\infty} n f_x(t)\mathrm{d}t = -\int_0^{n} t\mathrm{d}S_x(t) + n\,{}_np_x.$$

Then it follows again from integration by parts that

$$\mathring{e}_{x:\overline{n|}} = \int_0^{n} {}_tp_x\mathrm{d}t.$$

The following is a summary of the formulas for the moments of T_x.

Moments of T_x

(2.7) $$\mathring{e}_x = \int_0^\infty {}_tp_x \mathrm{d}t$$

(2.8) $$\mathrm{E}(T_x^2) = 2\int_0^\infty t\ {}_tp_x \mathrm{d}t$$

(2.9) $$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \mathrm{d}t$$

Example 2.14.

You are given $\mu_x = 0.01$ for all $x \geq 0$. Calculate the following:

(a) $\mathring{e}_x$

(b) $\mathrm{Var}(T_x)$

Solution:

(a) First of all, we have ${}_tp_x = e^{-0.01t}$. Then,

$$\mathring{e}_x = \int_0^\infty {}_tp_x\ \mathrm{d}t = \int_0^\infty e^{-0.01t}\ \mathrm{d}t = \frac{1}{-0.01}\left[e^{-0.01t}\right]_0^\infty = \frac{1}{0.01} = 100.$$

(b) We first calculate the second moment of T_x as follows:

$$\begin{aligned}\mathrm{E}(T_x^2) &= 2\int_0^\infty te^{-0.01t}\ \mathrm{d}t \\ &= \frac{-2}{0.01}\left(te^{-0.01t}\Big|_0^\infty - \int_0^\infty e^{-0.01t}\ \mathrm{d}t\right) \\ &= \frac{2}{0.01}\int_0^\infty e^{-0.01t}\ \mathrm{d}t \\ &= \frac{2}{0.01^2} = 20000.\end{aligned}$$

Then, the variance of T_x can be calculated as:

$$\begin{aligned}\mathrm{Var}(T_x) &= \mathrm{E}(T_x^2) - \left[\mathrm{E}(T_x)\right]^2 \\ &= 20000 - 100^2 \\ &= 10000.\end{aligned}$$

Alternatively, from $S_x(t) = e^{-0.01t}$, we see that $F_x(t) = 1 - e^{-0.01t}$. Since this is the cumulative distribution function of an exponential random variable with rate 0.01, it is immediate that T_x is exponentially distributed with rate 0.01. Hence, the mean is $1/0.01 = 100$ and the variance is $1/0.01^2 = 10000$. □

Example 2.15. **[Course 3 Fall 2001 #1]**

You are given:

$$\mu_x = \begin{cases} 0.04, & 0 < x < 40 \\ 0.05, & x > 40 \end{cases}$$

Calculate $\mathring{e}_{25:\overline{25}|}$.

(A) 14.0 (B) 14.4 (C) 14.8 (D) 15.2 (E) 15.6

Solution: First, we need to find ${}_tp_x$. Because the value of μ_x changes when x reaches 40, the derivation of ${}_tp_x$ is not as straightforward as that in the previous example.

For $0 < t < 15$, μ_{25+t} is always 0.04, and therefore

$${}_tp_{25} = \exp\left(-\int_0^t 0.04 \mathrm{d}u\right) = e^{-0.04t}.$$

For $t > 15$, μ_{25+t} becomes 0.05, and therefore

$${}_tp_{25} = {}_{15}p_{25} \times {}_{t-15}p_{40} = e^{-0.04\times 15} \exp\left(-\int_0^{t-15} 0.05 \mathrm{d}u\right) = e^{-0.04\times 15 - 0.05(t-15)} = e^{-0.05t+0.15}.$$

Given the expressions for ${}_tp_{25}$, we can calculate $\mathring{e}_{25:\overline{25}|}$ as follows:

$$\begin{aligned} \mathring{e}_{25:\overline{25}|} &= \int_0^{15} {}_tp_{25} \ \mathrm{d}t + \int_{15}^{25} {}_tp_{25} \ \mathrm{d}t \\ &= \int_0^{15} e^{-0.04t} \ \mathrm{d}t + \int_{15}^{25} e^{-0.05t+0.15} \ \mathrm{d}t \\ &= \left.\frac{e^{-0.04t}}{-0.04}\right|_0^{15} + e^{0.15}\left[\frac{e^{-0.05t}}{-0.05}\right]_{15}^{25} = 15.60. \end{aligned}$$

Hence, the answer is (E). □

Example 2.16. **[Structural Question SOA Sample #1]**

You are given the following survival function for a newborn:

$$S_0(t) = \frac{(121-t)^{1/2}}{k}, \quad 0 \le t \le \omega.$$

(a) Show that k must be 11 for $S_0(t)$ to be a valid survival function.

(b) Show that the limiting age, ω, for this survival model is 121.

(c) Calculate $\mathring{e}_0$ for this survival model.

(d) Derive an expression for μ_x for this survival model, simplifying the expression as much as possible.

(e) Calculate the probability, using the above survival model, that (57) dies between the ages of 84 and 100.

Solution:

(a) Recall that the first criterion for a valid survival function is that $S_0(0) = 1$. This implies that

$$\begin{aligned}\frac{(121-0)^{1/2}}{k} &= 1\\ (121)^{1/2} &= k\\ k &= 11\end{aligned}$$

(b) At the limiting age, the value of the survival function must be zero. Therefore,

$$\begin{aligned}S_0(\omega) &= 0\\ \frac{(121-\omega)^{1/2}}{k} &= 0\\ \omega &= 121\end{aligned}$$

(c) Using formula (2.8) with $x = 0$, we have

$$\begin{aligned}\mathring{e}_0 &= \int_0^{\omega-0} {}_tp_0 \,\mathrm{d}t\\ &= \int_0^{121} \frac{(121-t)^{1/2}}{11}\,\mathrm{d}t\\ &= \frac{1}{11}\left[\frac{-2}{3}(121-t)^{3/2}\right]_0^{121}\\ &= 80.6667\end{aligned}$$

(d) This part involves the relationship between the μ_x and $S_0(x)$, which was taught in Chapter 1:

$$\begin{aligned}\mu_x = -\frac{S_0'(x)}{S_0(x)} &= -\frac{\dfrac{\mathrm{d}}{\mathrm{d}x}\dfrac{(121-x)^{1/2}}{k}}{\dfrac{(121-x)^{1/2}}{k}}\\ &= -\frac{\dfrac{\mathrm{d}}{\mathrm{d}x}(121-x)^{1/2}}{(121-x)^{1/2}} = \frac{\dfrac{1}{2}(121-x)^{-1/2}}{(121-x)^{1/2}} = \frac{1}{2(121-x)}\end{aligned}$$

(e) First, we derive an expression for $S_{57}(t)$ as follows:

$$S_{57}(t) = \frac{S_0(57+t)}{S_0(57)} = \frac{\dfrac{(121-(57+t))^{1/2}}{11}}{\dfrac{(121-57)^{1/2}}{11}} = \sqrt{\frac{64-t}{64}}$$

The required probability is ${}_{27|16}q_{57}$, which can be calculated as follows:

$${}_{27|16}q_{57} = S_{57}(27) - S_{57}(43) = \sqrt{\frac{64-27}{64}} - \sqrt{\frac{64-43}{64}} = 0.1875$$

□

Now, we focus on the moments of the curtate future lifetime random variable K_x. The first moment of K_x is called the curtate expectation of life at age x, and is denoted by e_x. The formula for calculating e_x is derived as follows:

$$\begin{aligned} e_x &= \mathrm{E}(K_x) \\ &= \sum_{k=0}^{\infty} k\Pr(K_x = k) = \sum_{k=0}^{\infty} k \ {}_{k|}q_x \\ &= 0 \times q_x + 1 \times {}_{1|1}q_x + 2 \times {}_{2|1}q_x + 3 \times {}_{3|1}q_x + \dots \\ &= (p_x - {}_2p_x) + 2\left({}_2p_x - {}_3p_x\right) + 3\left({}_3p_x - {}_4p_x\right) + \dots \\ &= p_x + {}_2p_x + {}_3p_x + \dots \\ &= \sum_{k=1}^{\infty} {}_kp_x. \end{aligned}$$

If there is a limiting age, we replace ∞ with $\omega - x$.

The formula for calculating the second moment of K_x can be derived as follows:

$$\begin{aligned} \mathrm{E}(K_x^2) &= \sum_{k=0}^{\infty} k^2\Pr(K_x = k) = \sum_{k=0}^{\infty} k^2 \ {}_{k|}q_x \\ &= 0^2 \times q_x + 1^2 \times {}_{1|1}q_x + 2^2 \times {}_{2|1}q_x + 3^2 \times {}_{3|1}q_x + \dots \\ &= (p_x - {}_2p_x) + 4\left({}_2p_x - {}_3p_x\right) + 9\left({}_3p_x - {}_4p_x\right) + \dots \\ &= p_x + 3\ {}_2p_x + 5\ {}_3p_x + \dots \\ &= \sum_{k=1}^{\infty} (2k-1)_kp_x. \end{aligned}$$

Again, if there is a limiting age, we replace ∞ with $\omega - x$. Given the two formulas above, we can easily obtain $\mathrm{Var}(K_x)$.

In the exam, you may also be asked to calculate $\mathrm{E}(K_x \wedge n) = \mathrm{E}[\min(K_x, n)]$. This is called the n-year temporary curtate expectation of life at age x, and is denoted by $e_{x:\overline{n}|}$. It can be shown that

$$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x,$$

that is, instead of summing to infinity, we just sum to n.

There are two other equations that you need to know. First, you need to know that e_x and e_{x+1} are related to each other as follows:

$$e_x = p_x(1 + e_{x+1}).$$

Formulas of this form are called recursion formulas. We will further discuss recursion formulas in Chapters 3 and 4.

Second, assuming UDD holds, we have $T_x = K_x + U$, where U follows a uniform distribution over the interval [0,1]. Taking expectation on both sides, we have the following relation:

$$\mathring{e}_x = e_x + \frac{1}{2}.$$

The following is a summary of the key equations for the moments of K_x.

Moments of K_x

(2.10) $$e_x = \sum_{k=1}^{\infty} {}_kp_x$$

(2.11) $$\mathrm{E}(K_x^2) = \sum_{k=1}^{\infty} (2k-1)\,{}_kp_x$$

(2.12) $$e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x$$

(2.13) $$e_x = p_x(1 + e_{x+1})$$

(2.14) $$\text{Under UDD, } \mathring{e}_x = e_x + \frac{1}{2}$$

Example 2.17. You are given the following excerpt of a life table:

x	95	96	97	98
l_x	400	300	100	0

Calculate the following:

(a) e_{95}

(b) $\mathrm{Var}(K_{95})$

(c) $e_{95:\overline{1}|}$

(d) $\mathring{e}_{95}$, assuming UDD

(e) e_{96}, using the recursion formula

Solution:

(a) $e_{95} = \sum_{k=1}^{3} {}_kp_{95} = \frac{l_{96}}{l_{95}} + \frac{l_{97}}{l_{95}} + \frac{l_{98}}{l_{95}} = \frac{300}{400} + \frac{100}{400} + \frac{0}{400} = 1$

(b) We have

$$\mathrm{E}(K_{95}^2) = \sum_{k=1}^{3} (2k-1)\,{}_kp_{95} = \frac{l_{96}}{l_{95}} + \frac{3l_{97}}{l_{95}} + \frac{5l_{98}}{l_{95}} = \frac{300 + 3 \times 100 + 5 \times 0}{400} = 1.5.$$

Hence, $\mathrm{Var}(K_{95}) = 1.5 - 1^2 = 0.5$.

(c) $e_{95:\overline{1}|} = \sum_{k=1}^{1} {}_kp_{95} = \frac{l_{96}}{l_{95}} = \frac{300}{400} = 0.75.$

(d) Assuming UDD, $\mathring{e}_{95} = e_{95} + 0.5 = 1 + 0.5 = 1.5.$

(e) Using the recursion formula, $e_{95} = p_{95}\,(1 + e_{96}) = 0.75\,(1 + e_{96})$. Therefore,

$$e_{96} = 1/0.75 - 1 = 0.3333.$$

□

Example 2.18. **[MLC Fall 2012 #3]**

You are given:

(i) $S_0(t) = \left(1 - \frac{t}{\omega}\right)^{1/4}$, for $0 \le t \le \omega$

(ii) $\mu_{65} = 1/180$

Calculate e_{106}, the curtate expectation of life at age 106.

(A) 2.2 (B) 2.5 (C) 2.7 (D) 3.0 (E) 3.2

Solution: From statement (i), we know that there is a limiting age ω. Our first step is to compute the value of ω, using the information given.

Since

$$\mu_x = -\frac{\mathrm{d}}{\mathrm{d}x} \ln S_0(x) = \frac{1}{4(\omega - x)},$$

by statement (ii) we have

$$\frac{1}{4(\omega - 65)} = \frac{1}{180},$$

or

$$\omega = 110.$$

Then, using formula (2.9), we can calculate e_{106} as follows:

$$\begin{aligned} e_{106} &= p_{106} + {}_2p_{106} + {}_3p_{106} + {}_4p_{106} + \dots \\ &= \frac{S_0(107) + S_0(108) + S_0(109) + S_0(110) + \dots}{S_0(106)} \\ &= \frac{0.02727^{1/4} + 0.01818^{1/4} + 0.00090^{1/4} + 0 + \dots}{0.03636^{1/4}} \\ &= 2.4786 \end{aligned}$$

The answer is (B).

□

Example 2.19. **[MLC Spring 2012 #2]**

You are given:

(i) $p_x = 0.97$

(ii) $p_{x+1} = 0.95$

(iii) $e_{x+1.75} = 18.5$

(iv) Deaths are uniformly distributed between ages x and $x+1$.

(v) The force of mortality is constant between ages $x+1$ and $x+2$.

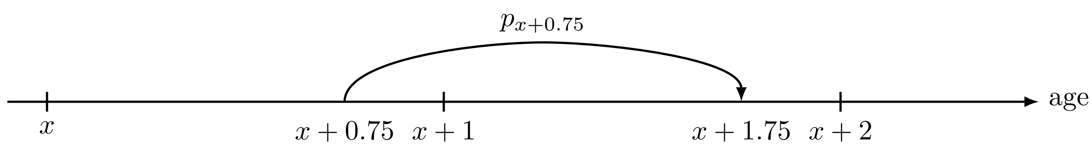
Calculate $e_{x+0.75}$.

(A) 18.6 (B) 18.8 (C) 19.0 (D) 19.2 (E) 19.4

Solution: Our goal is to calculate $e_{x+0.75}$. Since we are given the value of $e_{x+1.75}$, it is quite obvious that we should use the following recursive relation:

$$e_{x+0.75} = p_{x+0.75}\left(1 + e_{x+1.75}\right).$$

All then that remains is to calculate $p_{x+0.75}$. As shown in the following diagram, this survival probability covers part of the interval $[x, x+1)$ and part of the interval $[x+1, x+2)$.



We shall apply fractional age assumptions accordingly. Decomposing $p_{x+0.75}$, we have

$$p_{x+0.75} = {}_{0.25}p_{x+0.75} \times {}_{0.75}p_{x+1}.$$

According to statement (iv), the value of ${}_{0.25}p_{x+0.75}$ should be calculated by assuming UDD. Under this assumption, we have

$${}_{0.25}p_{x+0.75} = \frac{p_x}{{}_{0.75}p_x} = \frac{0.97}{1 - 0.75(1 - 0.97)} = 0.992327366.$$

According to statement (v), the value of ${}_{0.75}p_{x+1}$ should be calculated by assuming constant force of mortality over each year of age. Under this assumption, we have

$${}_{0.75}p_{x+1} = (p_{x+1})^{0.75} = (0.95)^{0.75} = 0.9622606.$$

It follows that $p_{x+0.75} = 0.992327366 \times 0.9622606 = 0.954878$.

Finally,

$$e_{x+0.75} = 0.954878 \times (1 + 18.5) = 18.620.$$

The answer is (A).

□

2.5 Useful Shortcuts

Constant Force of Mortality for All Ages

Very often, you are given that $\mu_x = \mu$ for all $x \geq 0$. In this case, we can easily find that

$$ {}_tp_x = e^{-\mu t}, \quad F_x(t) = 1 - e^{-\mu t}, \quad f_x(t) = \mu e^{-\mu t}. $$

From the density function, you can tell that in this case T_x follows an exponential distribution with parameter μ. By using the properties of an exponential distribution, we have

$$ \mathring{e}_x = \mathrm{E}(T_x) = 1/\mu, \quad \mathrm{Var}(T_x) = 1/\mu^2 \quad \text{for all } x. $$

These shortcuts can save you a lot of time on doing integration. For instance, had you known these shortcuts, you could complete Example 2.14 in a blink!

De Moivre's Law

De Moivre's law refers to the situation when

$$ l_x = \omega - x \text{ for } 0 \leq x < \omega, $$

or equivalently

$$ \mu_x = \frac{1}{\omega - x}. $$

De Moivre's law implies that the age at death random variable (T_0) is uniformly distributed over the interval $[0, \omega)$. It also implies that the future lifetime random variable (T_x) is uniformly distributed over the interval $[0, \omega - x)$, that is, for $0 \leq t < \omega - x$,

$$ {}_tp_x = 1 - \frac{t}{\omega - x}, \quad F_x(t) = \frac{t}{\omega - x}, \quad f_x(t) = \frac{1}{\omega - x}, \quad \mu_{x+t} = \frac{1}{\omega - x - t}. $$

By using the properties of uniform distributions, we can immediately obtain

$$ \mathring{e}_x = \frac{\omega - x}{2}, \quad \mathrm{Var}(T_x) = \frac{(\omega - x)^2}{12}. $$

The useful shortcuts are summarized in the following table.

Assumption	μ_{x+t}	${}_tp_x$	$F_x(t)$	$f_x(t)$	$\mathring{e}_x$	$\mathrm{Var}(T_x)$
Constant force for all ages	μ	$e^{-\mu t}$	$1 - e^{-\mu t}$	$\mu e^{-\mu t}$	$1/\mu$	$1/\mu^2$
De Moivre's law	$\frac{1}{\omega - x - t}$	$1 - \frac{t}{\omega - x}$	$\frac{t}{\omega - x}$	$\frac{1}{\omega - x}$	$\frac{\omega - x}{2}$	$\frac{(\omega - x)^2}{12}$

Example 2.20. You are given:

$$l_x = 100 - x, \quad 0 \le x < 100.$$

Calculate the following:

(a) ${}_{25}p_{25}$

(b) q_{25}

(c) μ_{50}

(d) $\mathring{e}_{50}$

Solution: First of all, note that $l_x = 100 - x$ for $0 \le x < 100$ means mortality follows De Moivre's law with $\omega = 100$.

(a) ${}_{25}p_{25} = 1 - \dfrac{25}{100-25} = \dfrac{2}{3}$.

(b) $q_{25} = 1 - l_{26}/l_{25} = 1 - 74/75 = 1/75$.
Alternatively, you can obtain the answer by using the fact that T_{25} is uniformly distributed over the interval [0,75). It immediately follows that the probability that (25) dies within one year is 1/75.

(c) $\mu_{50} = \dfrac{1}{100-50} = 0.02$.

(d) $\mathring{e}_{50} = \dfrac{100-50}{2} = 25$.

□

Example 2.21.

The survival function for the age-at-death random variable is given by

$$S_0(t) = 1 - \frac{t}{\omega}, \quad t \le \omega.$$

(a) Find an expression for $S_x(t)$, for $x < \omega$ and $t \le \omega - x$.

(b) Show that $\mu_x = \dfrac{1}{\omega - x}$, for $x < \omega$.

(c) Assuming $\mathring{e}_0 = 25$, show that $\omega = 50$.

Solution:

(a) The survival function implies that the age-at-death random variable is uniformly distributed over $[0, \omega]$. De Moivre's law applies here, so we can immediately write down the expression for $S_x(t)$ as follows:

$$S_x(t) = 1 - \frac{t}{\omega - x}.$$

(b) From Section 2.5, we know the expression for μ_x. However, since we are asked to prove the relation, we should show the steps involved:

$$\mu_x = -\frac{S_0'(x)}{S_0(x)} = -\frac{\frac{-1}{\omega}}{1-\frac{x}{\omega}} = \frac{1}{\omega - x}$$

(c) Under De Moivre's law, $\mathring{e}_0 = \frac{\omega}{2}$. Hence, we have $\frac{\omega}{2} = 25$, which gives $\omega = 50$.

□

Example 2.22.

It is given that $\mu_x = \mu$ for all $x \geq 0$.

(a) Show that $\mathring{e}_{x:\overline{n}|} = \dfrac{1-e^{-\mu n}}{\mu}$.

(b) Explain verbally why $\mathring{e}_{x:\overline{n}|}$ does not depend on x when we assume $\mu_x = \mu$ for all $x \geq 0$.

(c) State the value of $\mathring{e}_{x:\overline{n}|}$ when μ tends to zero. Explain your answer.

Solution:

(a) Since the force of mortality is constant for all ages, we have ${}_tp_x = e^{-\mu t}$. Then,

$$\mathring{e}_{x:\overline{n}|} = \int_0^n {}_tp_x \mathrm{d}t = \int_0^n e^{-\mu t}\mathrm{d}t = \frac{1-e^{-\mu n}}{\mu}.$$

(b) The assumption "$\mu_x = \mu$ for all $x \geq 0$" means that the future lifetime random variable is **exponentially distributed**. By the **memoryless property** of an exponential distribution, the expectation should be independent of the history (i.e., how long the life has survived).

(c) When μ tends to zero, $\mathring{e}_{x:\overline{n}|}$ tends to n. This is because when μ tends to zero, the underlying lives become immortal (i.e., the lives will live forever). As a result, the average number of years survived from age x to age $x+n$ (i.e., from time 0 to time n) must be n.

□

2.6 Exercise 2

1. You are given the following excerpt of a life table:

x	l_x
50	100,000
51	99,900
52	99,700
53	99,500
54	99,100
55	98,500

   Calculate the following:

   (a) ${}_2d_{52}$

   (b) ${}_{3|}q_{50}$

2. You are given:

$$l_x = 10000e^{-0.05x}, \quad x \geq 0.$$

   Find ${}_{5|15}q_{10}$.

3. You are given the following excerpt of a life table:

x	l_x
40	10,000
41	9,900
42	9,700
43	9,400
44	9,000
45	8,500

   Assuming uniform distribution of deaths between integral ages, calculate the following:

   (a) ${}_{0.2}p_{42}$

   (b) ${}_{2.6}q_{41}$

   (c) ${}_{1.6}q_{40.9}$

4. Repeat Question 3 by assuming constant force of mortality between integral ages.

5. You are given:

   (i) $l_{40} = 9{,}313{,}166$

   (ii) $l_{41} = 9{,}287{,}264$

   (iii) $l_{42} = 9{,}259{,}571$

   Assuming uniform distribution of deaths between integral ages, find ${}_{1.4}q_{40.3}$.

6. You are given:

x	l_x
40	60500
50	55800
60	50200
70	44000
80	36700

Assuming that deaths are uniformly distributed over each 10-year interval, find ${}_{15|20}q_{40}$.

7. You are given the following select-and-ultimate table with a select period of 2 years:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
50	0.02	0.04	0.06	52
51	0.03	0.05	0.07	53
52	0.04	0.06	0.08	54

Find ${}_{2|2}q_{[50]}$.

8. You are given the following select-and-ultimate table with a select period of 2 years:

x	$l_{[x]}$	$l_{[x]+1}$	l_{x+2}	$x+2$
70	22507	22200	21722	72
71	21500	21188	20696	73
72	20443	20126	19624	74
73	19339	19019	18508	75
74	18192	17871	17355	76

(a) Compute ${}_3p_{73}$.

(b) Compute the probability that a life age 71 dies between ages 75 and 76, given that the life was selected at age 70.

(c) Assuming uniform distribution of deaths between integral ages, calculate ${}_{0.5}p_{[70]+0.7}$.

(d) Assuming constant force of mortality between integral ages, calculate ${}_{0.5}p_{[70]+0.7}$.

9. You are given:

$$f_0(t) = \frac{20-t}{200}, \quad 0 \le t < 20.$$

Find $\mathring{e}_5$.

10. For a certain individual, you are given:

$$S_0(t) = \begin{cases} 1 - \dfrac{t}{100}, & 0 \le t < 30 \\ 0.7e^{-0.02(t-30)}, & t \ge 30 \end{cases}$$

Calculate $E(T_0)$ for the individual.

11. You are given:

$$\mu_x = \frac{2x}{400 - x^2}, \quad 0 \le x < 20.$$

Find $\text{Var}(T_0)$.

12. You are given:

(i) $\mu_x = \dfrac{1}{\omega - x}, \quad 0 \le x < \omega.$

(ii) $\text{Var}(T_0) = 468.75.$

Find ω.

13. You are given:

(i) $\mu_x = \mu$ for all $x \ge 0$.

(ii) $\mathring{e}_{30} = 40.$

Find ${}_5p_{20}$.

14. You are given:

$$l_x = 10000 - x^2, \quad 0 \le x \le 100.$$

Find $\text{Var}(T_0)$.

15. You are given:

$$\mu_x = 0.02, \quad x \ge 0.$$

Find $\mathring{e}_{10:\overline{10}|}$.

16. You are given:

(i) $S_0(t) = 1 - \dfrac{t}{\omega}, \quad 0 \le t < \omega.$

(ii) $\mathring{e}_{20:\overline{30}|} = 22.5.$

Calculate $\text{Var}(T_{30})$.

17. You are given:

$$l_x = 80 - x, \quad 0 \le x \le 80.$$

Find $\mathring{e}_{5:\overline{15}|}$.

18. (CAS, 2003 Fall #5) You are given:

(a) Mortality follows De Moivre's Law.

(b) $\mathring{e}_{20} = 30$.

Calculate q_{20}.

(A) 1/60 (B) 1/70 (C) 1/80 (D) 1/90 (E) 1/100

19. (2005 Nov #32) For a group of lives aged 30, containing an equal number of smokers and non-smokers, you are given:

(i) For non-smokers, $\mu_x^n = 0.08, x \ge 30$.

(ii) For smokers, $\mu_x^s = 0.16, x \ge 30$.

Calculate q_{80} for a life randomly selected from those surviving to age 80.

(A) 0.078 (B) 0.086 (C) 0.095 (D) 0.104 (E) 0.112

20. (2004 Nov #4) For a population which contains equal numbers of males and females at birth:

(i) For males: $\mu_x^m = 0.10, x \ge 0$.

(ii) For females: $\mu_x^f = 0.08, x \ge 0$.

Calculate q_{60} for this population.

(A) 0.076 (B) 0.081 (C) 0.086 (D) 0.091 (E) 0.096

21. (2000 May #1) You are given:

(i) $\mathring{e}_0 = 25$

(ii) $l_x = \omega - x, \quad 0 \le x \le \omega$.

(iii) T_x is the future lifetime random variable.

Calculate $\mathrm{Var}(T_{10})$.

(A) 65 (B) 93 (C) 133 (D) 178 (E) 333

22. (2005 May #21) You are given:

(i) $\mathring{e}_{30:\overline{40}|} = 27.692$

(ii) $S_0(t) = 1 - t/\omega, \quad 0 \le t \le \omega.$

(iii) T_x is the future lifetime random variable for (x).

Calculate $\mathrm{Var}(T_{30})$.

23. (2005 Nov # 13) The actuarial department for the SharpPoint Corporation models the lifetime of pencil sharpeners from purchase using a generalized De Moivre model with $S_0(t) = (1 - t/\omega)^\alpha$, for $\alpha > 0$ and $0 \le t \le \omega$.

A senior actuary examining mortality tables for pencil sharpeners has determined that the original value of α must change. You are given:

(i) The new complete expectation of life at purchase is half what it was previously.

(ii) The new force of mortality for pencil sharpeners is 2.25 times the previous force of mortality for all durations.

(iii) ω remains the same.

Calculate the original value of α.

(A) 1 (B) 2 (C) 3 (D) 4 (E) 5

24. (2000 Nov #25) You are given:

(i) Superscripts M and N identify two forces of mortality and the curtate expectations of life calculated from them.

(ii) $\mu^N_{25+t} = \begin{cases} \mu^M_{25+t} + 0.10(1-t), & 0 \le t \le 1 \\ \mu^M_{25+t}, & t > 1 \end{cases}$

(iii) $e^M_{25} = 10.0$

Calculate e^N_{25}.

(A) 9.2 (B) 9.3 (C) 9.4 (D) 9.5 (E) 9.6

25. (2003 Nov # 17) T_0, the future lifetime of (0), has a spliced distribution:

(i) $f^a(t)$ follows the Illustrative Life Table.

(ii) $f^b(t)$ follows De Moivre's law with $\omega = 100$.

(iii) The density function of T_0 is $f_0(t) = \begin{cases} kf^a(t), & 0 \le t \le 50 \\ 1.2f^b(t), & t > 50 \end{cases}$

Calculate ${}_{10}p_{40}$.

(A) 0.81 (B) 0.85 (C) 0.88 (D) 0.92 (E) 0.96

26. (a) Show that $e_x = p_x(1 + e_{x+1})$.

(b) Show that if deaths are uniformly distributed between integer ages, then

$$\mathring{e}_x = e_x + \frac{1}{2}.$$

(c) For a life table with a one-year select period, you are given:

x	$l_{[x]}$	$d_{[x]}$	l_{x+1}	$\mathring{e}_{[x]}$
80	1000	90	–	8.5
81	920	90	–	–

(i) Find l_{81} and l_{82}.

(ii) Assuming deaths are uniformly distributed over each year of age, calculate $\mathring{e}_{[81]}$.

27. For a certain group of individuals, you are given:

$$F_0(t) = 1 - e^{-0.02t}, \quad t \geq 0.$$

(a) Show that $S_x(t) = e^{-0.02t}$ for $x, t \geq 0$.

(b) Show that $\mu_x = 0.02$ for $x \geq 0$.

(c) Calculate $\mathring{e}_{10:\overline{10}|}$.

(d) Calculate e_{10}.

28. Consider the curtate future lifetime random variable, K_x.

(a) Explain verbally why $\Pr(K_x = k) = {}_{k|}q_x$ for $k = 0,1,\ldots$.

(b) Show that $e_{x:\overline{n}|} = \sum_{k=1}^{n} {}_kp_x$.

29. A mortality table is defined such that

$${}_tp_x = \left(1 - \frac{t}{100 - x}\right)^{0.5}$$

for $0 \leq x < 100$ and $0 \leq t < 100 - x$; and ${}_tp_x = 0$ for $t \geq 100 - x$.

(a) State the limiting age, ω.

(b) Calculate $\mathring{e}_{40}$.

(c) Calculate $\mathrm{Var}(T_{40})$.

30. (a) Define 'selection effect'.

(b) You are given the following two quotations for a 10-year term life insurance:

Company	**X**	**Y**
Policyholder	Age 28, non-smoker	Age 28, non-smoker
Medical exam required?	Yes	No
Annual premium	\$120.00	\$138.00

(i) Explain the difference between the two premiums in laymen's terms.

(ii) Explain the difference between the two premiums in actuarial terms.

(c) You are given the following select-and-ultimate life table:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
65	0.01	0.04	0.07	67
66	0.03	0.06	0.09	68
67	0.05	0.08	0.12	69

(i) State the select period.

(ii) Calculate ${}_{1|2}q_{[65]+1}$.

(iii) Calculate ${}_{0.4}p_{[66]+0.3}$, assuming constant force of mortality between integer ages.

31. You are given the following life table:

x	l_x	x	l_x	x	l_x
91	27	94	12	97	3
92	21	95	8	98	1
93	16	96	5	99	0

(a) Calculate e_{91}.

(b) Calculate $\mathring{e}_{91}$, assuming uniform distribution of deaths between integer ages.

32. You are given the following 4-year select-and-ultimate life table:

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	$q_{[x]+3}$	q_{x+4}	$x+4$
40	0.00101	0.00175	0.00205	0.00233	0.00257	44
41	0.00113	0.00188	0.00220	0.00252	0.00293	45
42	0.00127	0.00204	0.00240	0.00280	0.00337	46
43	0.00142	0.00220	0.00262	0.00316	0.00384	47
44	0.00157	0.00240	0.00301	0.00367	0.00445	48

(a) Calculate the index of selection at age 44, $I(44, k)$ for $k = 0,1,2,3$.

(b) Construct the table of $l_{[x]+t}$, for $x = 40, 41, 42$ and for all t. Use $l_{[40]} = 10{,}000$.

(c) Calculate the following probabilities:

(i) ${}_{2}p_{[42]}$

(ii) ${}_{3}q_{[41]+1}$

(iii) ${}_{3|2}q_{[41]}$

33. You are given the following excerpt of a life table:

x	l_x
50	100,000
51	99,900
52	99,700
53	99,500
54	99,100
55	98,500

(a) Calculate d_{52}.

(b) Calculate ${}_{2|}q_{50}$.

(c) Assuming uniform distribution of deaths between integer ages, calculate the value of ${}_{4.3}p_{50.4}$.

(d) Assuming constant force of mortality between integer ages, calculate the value of ${}_{4.3}p_{50.4}$.

2.7 Solutions to Exercise 2

1. (a) ${}_2d_{52} = l_{52} - l_{54} = 99700 - 99100 = 600.$

   (b) ${}_{3|}q_{50} = \dfrac{d_{53}}{l_{50}} = \dfrac{l_{53} - l_{54}}{l_{50}} = \dfrac{99500 - 99100}{100000} = 0.004.$

2. Expressing ${}_{5|15}q_{10}$ in terms of l_x, we have

$$\begin{aligned} {}_{5|15}q_{10} &= \frac{l_{15} - l_{30}}{l_{10}} \\ &= \frac{10000e^{-0.05\times 15} - 10000e^{-0.05\times 30}}{10000e^{-0.05\times 10}} \\ &= 0.4109. \end{aligned}$$

3. (a) ${}_{0.2}p_{42} = 1 - {}_{0.2}q_{42} = 1 - 0.2q_{42} = 1 - 0.2 \times (1 - 9400/9700) = 0.993814.$

   (b)
$$\begin{aligned} {}_{2.6}q_{41} &= 1 - {}_{2.6}p_{41} = 1 - {}_2p_{41} \times {}_{0.6}p_{43} = 1 - {}_2p_{41} \times (1 - 0.6q_{43}) \\ &= 1 - \frac{l_{43}}{l_{41}}\left(1 - 0.6 \times \frac{l_{43} - l_{44}}{l_{43}}\right) = 1 - \frac{9400}{9900}\left(1 - 0.6 \times \frac{9400 - 9000}{9400}\right) \\ &= 0.074747. \end{aligned}$$

   (c) Here, both subscripts are non-integers, so we need to use the trick. First, we compute ${}_{1.6}p_{40.9}$:

$${}_{0.9}p_{40} \times {}_{1.6}p_{40.9} = {}_{2.5}p_{40}$$

   Then, we have

$$\begin{aligned} {}_{1.6}p_{40.9} = \frac{{}_{2.5}p_{40}}{{}_{0.9}p_{40}} = \frac{{}_2p_{40}\ {}_{0.5}p_{42}}{{}_{0.9}p_{40}} = \frac{{}_2p_{40}\,(1 - 0.5q_{42})}{1 - 0.9q_{40}} &= \frac{\frac{9700}{10000}\left(1 - 0.5 \times \frac{300}{9700}\right)}{1 - 0.9 \times \frac{100}{10000}} \\ &= 0.963673. \end{aligned}$$

   Hence, ${}_{1.6}q_{40.9} = 1 - 0.963673 = 0.036327.$

4. (a) ${}_{0.2}p_{42} = (p_{42})^{0.2} = (9400/9700)^{0.2} = 0.993736.$

   (b)
$$\begin{aligned} {}_{2.6}q_{41} &= 1 - {}_{2.6}p_{41} = 1 - {}_2p_{41} \times {}_{0.6}p_{43} = 1 - {}_2p_{41} \times (p_{43})^{0.6} \\ &= 1 - \frac{l_{43}}{l_{41}}\left(\frac{l_{44}}{l_{43}}\right)^{0.6} = 1 - \frac{9400}{9900}\left(\frac{9000}{9400}\right)^{0.6} = 0.074958. \end{aligned}$$

   (c) First, we consider ${}_{1.6}p_{40.9}$:

$${}_{1.6}p_{40.9} = {}_{0.1}p_{40.9} \times {}_{1.5}p_{41} = {}_{0.1}p_{40.9} \times p_{41} \times {}_{0.5}p_{42} = (p_{40})^{0.1} \times p_{41} \times (p_{42})^{0.5}$$

   Hence,

$$\begin{aligned} {}_{1.6}q_{40.9} = 1 - (p_{40})^{0.1}(p_{41})(p_{42})^{0.5} &= 1 - \left(\frac{l_{41}}{l_{40}}\right)^{0.1}\left(\frac{l_{42}}{l_{41}}\right)\left(\frac{l_{43}}{l_{42}}\right)^{0.5} \\ &= 1 - \left(\frac{9900}{10000}\right)^{0.1}\left(\frac{9700}{9900}\right)\left(\frac{9400}{9700}\right)^{0.5} = 0.036441 \end{aligned}$$

5. By linear interpolation,

$$l_{40.3} = 0.7 \times l_{40} + 0.3 \times l_{41} = 0.7 \times 9{,}313{,}166 + 0.3 \times 9{,}287{,}264 = 9{,}305{,}395,$$

$$l_{41.7} = 0.3 \times l_{41} + 0.7 \times l_{42} = 0.3 \times 9{,}287{,}264 + 0.7 \times 9{,}259{,}571 = 9{,}267{,}879.$$

Therefore, ${}_{1.4}q_{40.3} = 1 - 9{,}267{,}879/9{,}305{,}395 = 0.004032$.

6. Expressing ${}_{15|20}q_{40}$ in terms of l_x, we have ${}_{15|20}q_{40} = \dfrac{l_{55} - l_{75}}{l_{40}}$.

From the table, we have $l_{40} = 60{,}500$. Since deaths are uniformly distributed over each 10-year span, we have

$$l_{55} = \frac{l_{50} + l_{60}}{2} = \frac{55{,}800 + 50{,}200}{2} = 53{,}000$$

$$l_{75} = \frac{l_{70} + l_{80}}{2} = \frac{44{,}000 + 36{,}700}{2} = 40{,}350$$

$${}_{15|20}q_{40} = \frac{53{,}000 - 40{,}350}{60{,}500} = 0.2091$$

7. ${}_{2|2}q_{[50]} = {}_2p_{[50]} - {}_4p_{[50]}$.

${}_2p_{[50]} = p_{[50]} \times p_{[50]+1} = 0.98 \times 0.96 = 0.9408$.

${}_4p_{[50]} = p_{[50]} \times p_{[50]+1} \times p_{52} \times p_{53} = 0.98 \times 0.96 \times 0.94 \times 0.93 = 0.8224$.

Hence, ${}_{2|2}q_{[50]} = 0.9408 - 0.8224 = 0.1184$.

8. (a) ${}_3p_{73} = l_{76}/l_{73} = 17{,}355/20{,}696 = 0.838568$.

(b) ${}_{4|}q_{[70]+1} = \dfrac{l_{[70]+5} - l_{[70]+6}}{l_{[70]+1}} = \dfrac{l_{75} - l_{76}}{l_{[70]+1}} = \frac{18{,}508-17{,}355}{22{,}200} = 0.05194$.

(c) By linear interpolation,

$$l_{[70]+0.7} = 0.3 \times l_{[70]} + 0.7 \times l_{[70]+1} = 0.3 \times 22{,}507 + 0.7 \times 22{,}200 = 22{,}292.1,$$

$$l_{[70]+1.2} = 0.8 \times l_{[70]+1} + 0.2 \times l_{[70]+2} = 0.8 \times 22{,}200 + 0.2 \times 21{,}722 = 22{,}104.4.$$

Hence, ${}_{0.5}p_{[70]+0.7} = 22{,}104.4/22{,}292.1 = 0.991580$.

(d)
$$\begin{aligned} {}_{0.5}p_{[70]+0.7} &= {}_{0.3}p_{[70]+0.7} \times {}_{0.2}p_{[70]+1} = (p_{[70]})^{0.3}(p_{[70]+1})^{0.2} \\ &= \left(\frac{l_{[70]+1}}{l_{[70]}}\right)^{0.3} \left(\frac{l_{72}}{l_{[70]+1}}\right)^{0.2} = 0.991562. \end{aligned}$$

9. We begin with finding $S_0(t)$ for $0 \le t \le 20$:

$$S_0(t) = \int_t^{20} f_0(u)\mathrm{d}u = \frac{\int_t^{20}(20-u)\mathrm{d}u}{200} = -\frac{\left[(20-u)^2\right]_t^{20}}{400} = \frac{(20-t)^2}{400}.$$

$$_tp_5 = \frac{S_0(t+5)}{S_0(5)} = \frac{(15-t)^2}{15^2} = \left(1-\frac{t}{15}\right)^2, \text{ for } 0 \le t \le 15.$$

$$\mathring{e}_5 = \int_0^{15} {}_tp_5 \,\mathrm{d}t = \int_0^{15}\left(1-\frac{t}{15}\right)^2 \mathrm{d}t = -\frac{15}{3}\left[\left(1-\frac{t}{15}\right)^3\right]_0^{15} = 5.$$

10. Since the survival function changes at $t = 30$, we need to decompose the integral into two parts.

$$\begin{aligned}
\mathrm{E}(T_0) &= \int_0^\infty S_0(t)\mathrm{d}t \\
&= \int_0^{30}\left(1-\frac{t}{100}\right)\mathrm{d}t + \int_{30}^\infty 0.7e^{-0.02(t-30)}\mathrm{d}t \\
&= \left[t - \frac{t^2}{200}\right]_0^{30} + 0.7\int_0^\infty e^{-0.02u}\mathrm{d}u \\
&= 25.5 + \frac{0.7}{0.02} \\
&= 60.5
\end{aligned}$$

11. We begin with the calculation of $_tp_0$:

$$_tp_0 = S_0(t) = e^{-\int_0^t \mu_u \mathrm{d}u} = e^{-\int_0^t \frac{2u}{400-u^2}\mathrm{d}u} = e^{\ln(400-u^2)|_0^t} = e^{\ln\left(\frac{400-t^2}{400}\right)} = 1 - \frac{t^2}{400}$$

$$\mathrm{E}(T_0) = \int_0^{20} {}_tp_0 \,\mathrm{d}t = \int_0^{20}\left(1-\frac{t^2}{400}\right)\mathrm{d}t = \left(t - \frac{t^3}{1200}\right)\Big|_0^{20} = \frac{40}{3}$$

$$\mathrm{E}(T_0^2) = 2\int_0^{20} t \,{}_tp_0 \,\mathrm{d}t = 2\int_0^{20}\left(t - \frac{t^3}{400}\right)\mathrm{d}t = 2\left(\frac{t^2}{2} - \frac{t^4}{1600}\right)\Big|_0^{20} = 200$$

$$\mathrm{Var}(T_0) = \mathrm{E}(T_0^2) - \left[\mathrm{E}(T_0)\right]^2 = 200 - \left(\frac{40}{3}\right)^2 = 22.22.$$

12. Here, the lifetime follows De Moivre's law (i.e., a uniform distribution). By using the properties of uniform distributions, we immediately obtain

$$\begin{aligned}
\mathrm{Var}(T_0) = 468.75 &= \frac{\omega^2}{12} \\
\omega^2 &= 5625 \\
\omega &= 75
\end{aligned}$$

13. When $\mu_x = \mu$ for all $x \geq 0$, the lifetime follows an exponential distribution. Using the properties of exponential distributions, we immediately obtain

$$\mathring{e}_{30} = 40 = \frac{1}{\mu} \quad \Rightarrow \quad \mu = \frac{1}{40} = 0.025.$$

Also, we know that when $\mu_x = \mu$ for all $x \geq 0, {}_tp_x = e^{-\mu t}$. Hence,

$${}_5p_{20} = e^{-0.025\times 5} = 0.8825.$$

14. First, we calculate $S_0(t)$

$$S_0(t) = \frac{l_t}{l_0} = \frac{10000 - t^2}{10000} = 1 - \frac{t^2}{10000}.$$

Then,

$$\mathrm{E}(T_0) = \int_0^{100} S_0(t)\mathrm{d}t = \int_0^{100} \left(1 - \frac{t^2}{10000}\right) \mathrm{d}t = \left(t - \frac{t^3}{30000}\right)\Bigg|_0^{100} = 66.6667,$$

and

$$\mathrm{E}(T_0^2) = 2\int_0^{100} tS_0(t)\mathrm{d}t = 2\int_0^{100} \left(t - \frac{t^3}{10000}\right) \mathrm{d}t = 2\left(\frac{t^2}{2} - \frac{t^4}{40000}\right)\Bigg|_0^{100} = 5000.$$

Hence, $\mathrm{Var}\,(T_0) = \mathrm{E}\left(T_0^2\right) - \left[\mathrm{E}\,(T_0)\right]^2 = 5000 - 66.6667^2 = 555.6.$

15. Since $\mu_x = 0.02$ for all $x \geq 0$, we immediately have ${}_tp_x = e^{-0.02t}$. Then,

$$\mathring{e}_{10:\overline{10}|} = \int_0^{10} {}_tp_{10}\ \mathrm{d}t = \int_0^{10} e^{-0.02t}\ \mathrm{d}t = -\left[\frac{1}{0.02}e^{-0.02t}\right]_0^{10} = 9.063.$$

16. First, we obtain ${}_tp_{20}$ as follows:

$${}_tp_{20} = \frac{S_0(20+t)}{S_0(20)} = 1 - \frac{t}{\omega - 20}.$$

Then, we have

$$\mathring{e}_{20:\overline{30}|} = \int_0^{30} {}_tp_{20}\ \mathrm{d}t = \int_0^{30} \left(1 - \frac{t}{\omega-20}\right) \mathrm{d}t = \left[t - \frac{t^2}{2(\omega-20)}\right]_0^{30} = 30 - \frac{450}{\omega - 20} = 22.5.$$

This gives $\omega = 80$.

Note that the underlying lifetime follows De Moivre's law. This implies that T_{30} is uniformly distributed over the interval $[0, \omega - 30)$, that is, [0,50). Using the properties of uniform distributions, we have $\mathrm{Var}\,(T_{30}) = 50^2/12 = 208.33$.

17. First, we compute ${}_tp_5$:

$${}_tp_5 = \frac{l_{5+t}}{l_5} = \frac{80 - (5+t)}{80-5} = 1 - \frac{t}{75}.$$

Hence,

$$\mathring{e}_{5:\overline{15}|} = \int_0^{15} \left(1 - \frac{t}{75}\right) \mathrm{d}t = \left[t - \frac{t^2}{150}\right]_0^{15} = 13.5.$$

18. Since mortality follows De Moivre's law, for 20 year olds, future lifetime follows a uniform distribution over $[0, \omega - 20)$. We have

$$\mathring{e}_{20} = 30 = \frac{\omega - 20}{2},$$

which gives $\omega = 80$. Since death occurs uniformly over [0,60), we have $q_{20} = 1/60$. Hence, the answer is (A).

19. The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 80.

- Suppose that there are l_{30} persons in the entire population initially. At time 0 (i.e., at age 30), there are $0.5l_{30}$ nonsmokers and $0.5l_{30}$ smokers.
- For nonsmokers, the proportion of individuals who can survive to age 80 is $e^{-0.08\times 50} = e^{-4}$. For smokers, the proportion of individuals who can survive to age 80 is $e^{-0.16\times 50} = e^{-8}$. As a result, at age 80, there are $0.5l_{30}e^{-4}$ nonsmokers and $0.5l_{30}e^{-0.8}$ smokers. Hence, among those who can survive to age 80,
$$\frac{0.5l_{30}e^{-4}}{0.5l_{30}e^{-4} + 0.5l_{30}e^{-8}} = \frac{1}{1+e^{-4}} = 0.982014$$
are nonsmokers and $1 - 0.982014 = 0.017986$ are smokers.

Second, we need to calculate q_{80} for both smokers and nonsmokers.

- For a nonsmoker at age 80, $q^n_{80} = 1 - e^{-0.08}$.
- For a smoker at age 80, $q^s_{80} = 1 - e^{-0.16}$.

Finally, for the whole population, we have

$$q_{80} = 0.982014(1 - e^{-0.08}) + 0.017986(1 - e^{-0.16}) = 0.07816.$$

Hence, the answer is (A).

20. The calculation of the required probability involves two steps.

First, we need to know the composition of the population at age 60.

- Suppose that there are l_0 persons in the entire population initially. At time 0 (i.e., at age 0), there are $0.5l_0$ males and $0.5l_0$ females.
- For males, the proportion of individuals who can survive to age 60 is $e^{-0.10\times 60} = e^{-6}$. For females, the proportion of individuals who can survive to age 60 is $e^{-0.08\times 60} = e^{-4.8}$. As a result, at age 60, there are $0.5l_0e^{-6}$ males and $0.5l_0e^{-4.8}$ females. Hence, among those who can survive to age 60,
$$\frac{0.5l_0e^{-6}}{0.5l_0e^{-6} + 0.5l_0e^{-4.8}} = \frac{1}{1+e^{1.2}} = 0.231475$$
are males and $1 - 0.231475 = 0.768525$ are females.

Second, we need to calculate q_{60} for both males and females.

- For a male at age 60, $q_{60}^m = 1 - e^{-0.10}$.
- For a female at age 60, $q_{60}^f = 1 - e^{-0.08}$.

Finally, for the whole population, we have

$$q_{60} = 0.231475(1 - e^{-0.10}) + 0.768525(1 - e^{-0.08}) = 0.0811.$$

Hence, the answer is (B).

21. From Statement (ii), we know that the underlying lifetime follows De Moivre's law. By using the properties of uniform distributions, we immediately have

$$\mathring{e}_0 = \frac{\omega}{2} = 25,$$

which gives $\omega = 50$.

Under De Moivre's law, T_{10} is uniformly distributed over the interval $[0, \omega - 10)$, that is, [0,40). By using the properties of uniform distributions, we immediately obtain

$$\text{Var}(T_{10}) = 40^2/12 = 133.3.$$

Hence, the answer is (C).

22. From the given survival function, we know that the underlying lifetime follows De Moivre's law. First, we find ${}_tp_{30}$:

$${}_tp_{30} = \frac{S_0(30+t)}{S_0(30)} = 1 - \frac{t}{\omega - 30}.$$

We then use Statement (i) to find ω:

$$\mathring{e}_{30:\overline{40}|} = \int_0^{40} {}_tp_{30}\,\text{d}t = \int_0^{40} \left(1 - \frac{t}{\omega - 30}\right) \text{d}t = \left[t - \frac{t^2}{2(\omega - 30)}\right]_0^{40}$$
$$= 40 - \frac{40^2}{2(\omega - 30)} = 27.692.$$

This gives $\omega = 95$.

Under De Moivre's law, T_{30} is uniformly distributed over $[0, \omega - 30)$, that is [0,65). By using the properties of uniform distributions, we immediately obtain

$$\text{Var}(T_{30}) = 65^2/12 = 352.1.$$

Hence the answer is (B).

23. For the original model, $S_0(t) = (1 - t/\omega)^\alpha$. This gives

$$\text{E}(T_0) = \int_0^\omega S_0(t)\text{d}t = \int_0^\omega \left(1 - \frac{t}{\omega}\right)^\alpha \text{d}t = -\frac{\omega}{\alpha + 1}\left(1 - \frac{t}{\omega}\right)^{\alpha+1}\Bigg|_0^\omega = \frac{\omega}{\alpha+1},$$

and

$$\mu_x = -\frac{S_0'(x)}{S_0(x)} = \frac{\frac{\alpha}{\omega}\left(1 - \frac{x}{\omega}\right)^{\alpha - 1}}{\left(1 - \frac{x}{\omega}\right)^\alpha} = \frac{\alpha}{\omega - x}.$$

Let α and α^* be the original and new values of α, respectively. Since the new complete expectation of life is half what it was previously, we have

$$\frac{\omega}{\alpha^*+1} = \frac{1}{2}\left(\frac{\omega}{\alpha+1}\right), \text{ or } 2(\alpha+1) = \alpha^*+1.$$

Also, since the new force of mortality is 2.25 times the previous force of mortality for all durations, we have

$$\frac{\alpha^*}{\omega - x} = \frac{2.25\alpha}{\omega - x},$$

or $\alpha^* = 2.25\alpha$. Solving $2(\alpha+1) = 2.25\alpha + 1$, we obtain $\alpha = 4$. Hence, the answer is (D).

24. The primary objective of this question is to examine your knowledge on the recursion formula $e_x = p_x(1 + e_{x+1})$.

Note that M and N have the same force of mortality from age 26. This means that

$${}_kp_{26}^M = {}_kp_{26}^N, \quad k = 1,2,3,\ldots,$$

and consequently that

$$e_{26}^M = e_{26}^N.$$

Using the identity above, we have

$$e_{25}^N = p_{25}^N(1 + e_{26}^N) = p_{25}^N(1 + e_{26}^M).$$

We can find p_{25}^N using the force of mortality given:

$$\begin{aligned}
p_{25}^N &= \exp\left(-\int_0^1 \mu_{25+t}^N \mathrm{d}t\right) \\
&= \exp\left(-\int_0^1 (\mu_{25+t}^M + 0.1(1-t))\mathrm{d}t\right) \\
&= e^{-\int_0^1 \mu_{25+t}^M \mathrm{d}t} e^{-\int_0^1 0.1(1-t)\mathrm{d}t} \\
&= p_{25}^M \exp\left([0.05(1-t)^2]_0^1\right) \\
&= p_{25}^M e^{-0.05}.
\end{aligned}$$

This implies that

$$e_{25}^N = e^{-0.05} p_{25}^M (1 + e_{26}^M) = e^{-0.05} e_{25}^M = 0.951 \times 10 = 9.51.$$

Hence, the answer is (D).

25. Splicing two functions $h(x)$ and $g(x)$ on an interval $[a,b]$ means that we break up the interval into two smaller intervals $[a,c]$ and $(c,d]$ and define the spliced function to equal $h(x)$ on $[a,c]$ and $g(x)$ on $(c,d]$. In this case, we are breaking up [0,100] into [0,50] and (50,100]. Our new function will equal $kf^a(t)$ on [0,50] and $1.2f^b(t)$ on (50, 100]. The spliced function needs to be a density function on [0, 100], so we need to find the value of k that makes the total area under the curve equal 1.

We will start by looking at $1.2f^b(t)$. For a De Moivre's model with $\omega = 100$, $f^b(t) = 1/100$ which means $1.2f^b(t) = 1.2/100$. Thus, the area under the curve (50,100] is

$$\int_{50}^{100} \frac{1.2}{100}\,\mathrm{d}t = \frac{1.2(100-50)}{100} = 0.6.$$

This means that the area under the curve on [0, 50] must be 0.4. So,

$$k\int_0^{50} f^a(t)\mathrm{d}t = k\left(\frac{l_0 - l_{50}}{l_0}\right) = k\frac{1049099}{10000000} = 0.4,$$

where l_t is the life function that corresponds to the Illustrative Life Table. This gives $k = 3.8128$.

Let ${}_tq_0{}^*$ be death probabilities that corresponds to the Illustrative Life Table. Then

$$\begin{aligned}
{}_{10}p_{40} =& \frac{{}_{50}p_0}{{}_{40}p_0} = \frac{1 - {}_{50}q_0}{1 - {}_{40}q_0} \\
=& \frac{1 - \int_0^{50} kf^a(t)\mathrm{d}t}{1 - \int_0^{40} kf^a(t)\mathrm{d}t} = \frac{1 - k\ {}_{50}q_0{}^*}{1 - k\ {}_{40}q_0{}^*} \\
=& \frac{1 - 0.4}{1 - k\dfrac{686834}{10000000}} = \frac{0.6}{0.738124} = 0.8129.
\end{aligned}$$

Hence, the answer is (A).

26. (a) The proof is as follows:

$$\begin{aligned}
e_x &= \sum_{k=1}^{\infty} {}_kp_x = p_x + \sum_{k=2}^{\infty} {}_kp_x \\
&= p_x + p_x \sum_{k=2}^{\infty} {}_{k-1}p_{x+1} = p_x + p_x \sum_{j=1}^{\infty} p_{x+1} \\
&= p_x + p_x e_{x+1} = p_x\left(1 + e_{x+1}\right)
\end{aligned}$$

(b) Under UDD, $T_x = K_x + U$, where U follows a uniform distribution over the interval [0, 1]. Taking expectation on both sides, we have $\mathrm{E}(T_x) = \mathrm{E}(K_x) + \mathrm{E}(U)$, which implies

$$\mathring{e}_x = e_x + \frac{1}{2}.$$

(c) This is a difficult question. To answer this question, you need to use the following three facts:

- For a one-year select period, $l_{[x]+1} = l_{x+1} = l_{[x]} - d_{[x]}$ and $e_{[x]+1} = e_{x+1}$.
- $e_x = p_x(1 + e_{x+1})$
- Under UDD, $\mathring{e}_x = e_x + \dfrac{1}{2}$

We can complete the second last column of the table by using $l_{[x]+1} = l_{x+1} = l_{[x]} - d_{[x]}$:

$$l_{81} = l_{[80]} - d_{[80]} = 1000 - 90 = 910,$$

$$l_{82} = l_{[81]} - d_{[81]} = 920 - 90 = 830.$$

Under UDD, we have $e_{[80]} = 8.5 - 0.5 = 8$.

We then apply the recursion formula as follows:

$$e_{[80]} = p_{[80]}(1 + e_{[80]+1}) = p_{[80]}\left(1 + e_{81}\right) = \frac{910}{1000}\left(1 + e_{81}\right).$$

This gives $e_{81} = 7.791208791$.

Also, we have

$$e_{[81]} = p_{[81]}(1 + e_{82}), \quad e_{81} = p_{81}(1 + e_{82}).$$

This means

$$\begin{aligned} e_{[81]} &= e_{81}\frac{p_{[81]}}{p_{81}} = e_{81}\frac{l_{[81]+1}/l_{[81]}}{l_{82}/l_{81}} = e_{81}\frac{l_{82}/l_{[81]}}{l_{82}/l_{81}} \\ &= e_{81}\frac{l_{81}}{l_{[81]}} = 7.791208791 \times \frac{910}{920} = 7.7065. \end{aligned}$$

Finally, assuming UDD, $\mathring{e}_{[81]} = 7.7065 + 0.5 = 8.2065$.

27. (a) $S_x(t) = \dfrac{S_0(x+t)}{S_0(x)} = \dfrac{e^{-0.02(x+t)}}{e^{-0.02x}} = e^{-0.02t}$

(b) $\mu_x = \dfrac{-S_0'(x)}{S_0(x)} = \dfrac{0.02e^{-0.02x}}{e^{-0.02x}} = 0.02$

(c) $\mathring{e}_{10:\overline{10}|} = \displaystyle\int_0^{10} {}_tp_{10}\mathrm{d}t = \int_0^{10} e^{-0.02t}\mathrm{d}t = \left.\frac{e^{-0.02t}}{-0.02}\right|_0^{10} = 9.06346$

(d) Since ${}_kp_x = S_x(k) = e^{-0.02k}$, we have

$$e_x = \sum_{k=1}^{\infty} {}_kp_x = \sum_{k=1}^{\infty} e^{-0.02k} = \frac{e^{-0.02}}{1 - e^{-0.02}} = 49.5017.$$

28. (a) The event $K_x = k$ is the same as $k \le T_x < k + 1$, which means the individual cannot die within the first k years and must die during the subsequent year. The probability associated with this event must be ${}_{k|}q_x$, the k-year deferred one-year death probability.

(b) The proof is as follows:

$$\begin{aligned} e_{x:\overline{n}|} &= \sum_{k=0}^{n-1} k\Pr\left(K_x = k\right) + \sum_{k=n}^{\infty} n\Pr\left(K_x = k\right) \\ &= \sum_{k=0}^{n-1} k \times {}_{k|}q_x + \sum_{k=n}^{\infty} n \times {}_{k|}q_x \\ &= 0 + 1 \times {}_{1|}q_x + 2 \times {}_{2|}q_x + \ldots + (n-1) \times {}_{n-1|}q_x \\ &\qquad + n \times {}_{n|}q_x + n \times {}_{n+1}q_x + \ldots \\ &= (p_x - {}_2p_x) + 2\left({}_2p_x - {}_3p_x\right) + \ldots + (n-1)\left({}_{n-1}p_x - {}_np_x\right) + \ldots \\ &\qquad + n\left({}_np_x - {}_{n+1}p_x\right) + n\left({}_{n+1}p_x - {}_{n+2}p_x\right) + n\left({}_{n+2}p_x - {}_{n+3}p_x\right) + \ldots \\ &= p_x + {}_2p_x + {}_3p_x + \ldots + {}_np_x \\ &= \sum_{k=1}^{n} {}_kp_x \end{aligned}$$

29. (a) $\omega = 100$

(b) $\mathring{e}_{40} = \int_0^{100-40} {}_tp_{40}\,\mathrm{d}t = \int_0^{60}\left(1-\frac{t}{60}\right)^{0.5}\mathrm{d}t$

$$= \frac{-60}{1.5}\left(1-\frac{t}{60}\right)^{1.5}\Bigg|_0^{60} = 0 + \frac{60}{1.5} = 40$$

(c) $\mathrm{E}(T_{40}^2) = 2\int_0^{60} t\left(1-\frac{t}{60}\right)^{0.5}\mathrm{d}t.$

Let $y = 1 - t/60$. We have

$$\begin{aligned}\mathrm{E}(T_{40}^2) &= 2\int_0^{60} 60(1-y)y^{0.5}(-60)\mathrm{d}y\\ &= 7200\int_0^1 \left(y^{1/2}-y^{3/2}\right)\mathrm{d}y\\ &= 7200\left[\frac{2}{3}y^{3/2}-\frac{2}{5}y^{5/2}\right]_0^1 = 1920\end{aligned}$$

Hence, $\mathrm{Var}(T_{40}) = 1920 - 40^2 = 320$.

30. (a) The effect of medical (or other) evidence at the inception of an insurance contract.

(b) (i) In laymen's terms:

- Company Y requires no medical examination, so it is taking more risk.
- Company Y has a higher change of adverse selection.
- Company Y has to charge more premium to compensate for the additional risk.

(ii) In actuarial terms:

- Company X requires a medical examination, which means there is a stronger effect of selection.
- The index of selection is higher (closer to 1).
- The death probabilities used to price the policy are lower. This means the premium charged by Company X is lower than that charged by Company Y.

(c) (i) 2 years.

(ii) $$\begin{aligned}{}_{1|2}q_{[65]+1} &= p_{[65]+1}\times {}_2q_{[65]+2}\\ &= p_{[65]+1}\times {}_2q_{67}\\ &= p_{[65]+1}\left[1-(1-q_{67})(1-q_{68})\right]\\ &= (1-0.04)[1-(1-0.07)(1-0.09)] = 0.147552.\end{aligned}$$

(iii) ${}_{0.4}p_{[66]+0.3} = (p_{[66]})^{0.4} = (1-0.03)^{0.4} = 0.987890.$

31. (a) $e_{91} = p_{91} + {}_2p_{91} + {}_3p_{91} + \ldots = \left(l_{92}+l_{93}+\ldots+l_{99}\right)/l_{91} = 2.44$ years.

(b) Under UDD $\mathring{e}_{91} = e_{91} + 0.5 = 2.94$.

32. (a) $I(44{,}0) = 1 - \dfrac{q_{[44]}}{q_{44}} = 1 - \dfrac{0.00157}{0.00257} = 0.38911$

$I(44{,}1) = 1 - \dfrac{q_{[44]+1}}{q_{44+1}} = 1 - \dfrac{0.00240}{0.00293} = 0.18089$

$I(44{,}2) = 1 - \dfrac{q_{[44]+2}}{q_{44+2}} = 1 - \dfrac{0.00301}{0.00337} = 0.10682$

$I(44{,}3) = 1 - \dfrac{q_{[44]+3}}{q_{44+3}} = 1 - \dfrac{0.00367}{0.00384} = 0.04427$

Comment: The index of selection reduces as the value of k increases. This agrees with the fact that as duration increases, selection effect tapers off.

(b) The table is calculated as follows:

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}	$x+4$
40	10000	9989.9	9972.4	9952.0	9928.8	44
41	9980.2	9969.0	9950.2	9928.3	9903.3	45
42	9958.8	9946.1	9925.8	9902.0	9987.3	46

(c) (i) ${}_2p_{[42]} = l_{[42]+2}/l_{[42]} = 0.99669$

(ii) ${}_3q_{[41]+1} = (l_{[41]+1} - l_{[41]+4})/l_{[41]+1} = 0.00659$

(iii) ${}_{3|2}q_{[41]} = (l_{[41]+3} - l_{[41]+5)})/l_{[41]} = 0.00542.$

33. (a) $d_{52} = l_{52} - l_{53} = 99700 - 99500 = 200$

(b) ${}_{2|}q_{50} = {}_2p_{50}q_{52} = \dfrac{l_{52}}{l_{50}}\dfrac{d_{52}}{l_{52}} = \dfrac{d_{52}}{l_{50}} = \dfrac{200}{100000} = 0.002$

(c) Since ${}_{0.4}p_{50}\ {}_{4.3}p_{50.4} = {}_{4.7}p_{50}$, we have

$$ {}_{4.3}p_{50.4} = \frac{{}_{4.7}p_{50}}{{}_{0.4}p_{50}} = \frac{{}_4p_{50} \times {}_{0.7}p_{54}}{{}_{0.4}p_{50}} = \frac{{}_4p_{50}\,(1 - 0.7q_{54})}{1 - 0.4q_{50}} = \frac{\frac{l_{54}}{l_{50}}\left(1 - 0.7\frac{d_{54}}{l_{54}}\right)}{1 - 0.4\frac{d_{50}}{l_{50}}}. $$

Substituting, we obtain ${}_{4.3}p_{50.4} = 0.987195.$

(d) $$ \begin{aligned} {}_{4.3}p_{50.4} &= {}_{0.6}p_{50} \times {}_{3.7}p_{51} = {}_{0.6}p_{50.4} \times {}_3p_{51} \times {}_{0.7}p_{54} \\ &= (p_{50})^{0.6}\ {}_3p_{51}\ (p_{54})^{0.7} \\ &= \left(\frac{99900}{100000}\right)^{0.6}\left(\frac{99100}{99900}\right)\left(\frac{98500}{99100}\right)^{0.7} = 0.987191 \end{aligned} $$

Mock Tests

There are 34 multiple choice questions on Exam FAM and the time limit is 3.5 hours. In each of the five mock tests, there are 17 questions and you should aim at spending 1 hr 45 minutes on it.

Mock Test 1

****BEGINNING OF EXAMINATION****

1. For a whole life insurance of 1,000 issued to life selected at age x,

   (i) Percent of premium expenses is 90% in the first year, and 10% in each year thereafter.

   (ii) Maintenance expenses are 15 per 1,000 of insurance in the first year, and 3 per 1,000 of insurance thereafter.

   (iii) Claim settlement expenses are 10 per 1,000 of insurance.

   (iv) A 15-year select and ultimate mortality is to be used.

   Determine the expression of the gross premium for the policy using equivalence principle.

   (A) $\dfrac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.9}$

   (B) $\dfrac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}$

   (C) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{\ddot{a}_{[x]} - 0.9}$

   (D) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.9}$

   (E) $\dfrac{1010\bar{A}_{[x]} + 15 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8}$

2. You are given:

   (i) $i = 0.07$

   (ii) $\ddot{a}_x = 11.7089$

   (iii) $\ddot{a}_{x:\overline{40}|} = 11.55$

   (iv) $\ddot{a}_{x+40} = 7.1889$

   Find $A^{1}_{x:\overline{40}|}$.

   (A) 0.120 (B) 0.147 (C) 0.172 (D) 0.197 (E) 0.222

3. You are given the following select-and-ultimate table:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
65	0.11	0.13	0.15	67
66	0.12	0.135	0.16	68
67	0.13	0.145	0.17	69

   Deaths are uniformly distributed over each year of age.

   Find ${}_{1.8}p_{[66]+0.6}$.

   (A) 0.69 (B) 0.71 (C) 0.73 (D) 0.75 (E) 0.77

4. For a 3-year fully discrete endowment insurance of 1,000 on (x), you are given:

(i) $q_x = 0.1$

(ii) $q_{x+1} = 0.15$

(iii) $v = 0.9$

(iv) Deaths are uniformly distributed over each year of age.

Calculate the net premium policy value 9 months after the issuance of the policy.

(A) 274 (B) 280 (C) 283 (D) 286 (E) 290

5. Which of the following statements is/are correct?

I. Insurable interest in an entity exists if one would suffer a financial loss if that entity is damaged.

II. Insurable interest is related to the concept of adverse selection.

III. Stranger owned life insurance is illegal in many jurisdiction because the purchaser has no insurable interest in the insured.

(A) I only (B) II only (C) I and III only
(D) II and III only (E) I, II and III

6. Which of the following is/are strictly increasing function(s) of T_x for all $T_x \geq 0$?

I. The present value random variable for a continuous whole life annuity of \$1 on (x)

II. The present value random variable for a continuous n-year temporary life annuity of \$1 on (x)

III. The net future loss at issue random variable for a fully continuous whole life insurance of \$1 on (x)

(A) I only (B) III only (C) I, II only
(D) I, III only (E) I, II and III

7. For a fully discrete whole life insurance on (30), you are given:

(i) $i = 0.05$

(ii) $q_{29+h} = 0.004$

(iii) The net amount at risk for policy year h is 1295.

(iv) The terminal policy value for policy year $h-1$ is 179.

(v) $\ddot{a}_{30} = 16.2$

Calculate the initial policy value for policy year $h+1$.

(A) 188 (B) 192 (C) 200 (D) 214 (E) 226

8. You are given:

(i) $\mu_{x+t} = \begin{cases} 0.02 & 0 \le t < 1 \\ 0.07 & 1 \le t < 2 \end{cases}$ (ii) $Y = \min(T_x, 2)$

Calculate $E(Y)$.

(A) 1.88 (B) 1.90 (C) 1.92 (D) 1.94 (E) 1.96

9. An insurer issues fully discrete whole life insurance policies to a group of 50 high-risk drivers all aged 35 with a sum insured of 2,000. You are given:

(i) The mortality of each driver follows the Standard Ultimate Life Table with an age rating of 3 years. That is,

$$q_x = q_{x+3}^{SULT},$$

where q_y^{SULT} is the 1-year death probability under the Standard Ultimate Life Table.

(ii) Lifetimes of the group of 50 high-risk drivers are independent.

(iii) Premiums are payable annually in advance. Each premium is 110% of the net annual premium.

(iv) $i = 0.05$

By assuming a normal approximation, estimate the 95^{th}-percentile of the net future loss random variable.

(A) 1090 (B) 1240 (C) 1390 (D) 1540 (E) 1690

10. Which of the following regarding is/are correct?

I. Variable annuity has cash value.

II. Term insurance has cash value.

III. Universal life insurance has cash value.

(A) I only (B) II only (C) III only
(D) II and III only (E) I, II and III

11. You are given:

(i) $\delta = 0.05$

(ii) $\bar{A}_x = 0.44$

(iii) ${}^2\bar{A}_x = 0.22$

Consider a portfolio of 100 fully continuous whole life insurances. The ages of the all insureds are x, and their lifetimes are independent. The face amount of the policies, the premium rate and the number of policies are as follows:

Face amount	Premium rate	Number of Policies
100	4.3	75
400	17.5	25

By using a normal approximation, calculate the probability that the present value of the aggregate loss-at-issue for the insurer's portfolio will exceed 700.

(A) $1 - \Phi(2.28)$ (B) $1 - \Phi(0.17)$ (C) $\Phi(0)$

(D) $\Phi(0.17)$ (E) $\Phi(2.28)$

12. Victor is now age 22, and his future lifetime has the following cumulative distribution function:

$$F_{22}(t) = 1 - (1 + 0.04t)e^{-0.04t}.$$

Let Z be the present value random variable for a fully continuous life insurance that pays 100 immediately on the death of Victor provided that he dies between ages 32 and 52.

The force of interest is 0.06.

Find the 70^{th}-percentile of Z.

(A) 0 (B) 25 (C) 40 (D) 47 (E) 50

13. You are given:

(i) $i = 0.05$

(ii) $A_{60} = 0.560$

(iii) $A_{40} = 0.176$

(iv) ${}_{20}p_{40} = 0.75$

Calculate $a^{(3)}_{40:\overline{20}|}$ using the two-term Woolhouse's approximation.

(A) 13.7 (B) 14.2 (C) 14.5 (D) 14.9 (E) 15.1

14. You are given:

(i) $\delta = 0.05$

(ii) $\ddot{a}_{60} = 12.18$

(iii) $p_{60} = 0.98$

Using the claims accelerated approach, calculate $\ddot{a}_{61}^{(6)}$.

(A) 11.6 (B) 11.7 (C) 11.8 (D) 11.9 (E) 12.1

15. Let Y be the present value random variable for a special three-year temporary life annuity on (x). You are given:

(i) The life annuity pays $2 + k$ at time k, for $k = 0$, 1 and 2.

(ii) $v = 0.9$

(iii) $p_x = 0.8$, $p_{x+1} = 0.75$, $p_{x+2} = 0.5$

Calculate the standard deviation of Y.

(A) 1.2 (B) 1.8 (C) 2.4 (D) 3.0 (E) 3.6

16. You are given:

$$\mu_x = \begin{cases} 0.04 & 50 \leq x < 60 \\ 0.05 + 0.001(x-60)^2 & 60 \leq x < 70 \end{cases}$$

Calculate ${}_{4|14}q_{50}$.

(A) 0.38 (B) 0.44 (C) 0.47 (D) 0.50 (E) 0.56

17. You are given:

(i) $p_{41} = 0.999422$

(ii) $i = 0.03$

(iii) $\ddot{a}_{42:\overline{23}|} = 16.7147$

Calculate the Full Preliminary Term reserve at time 2 for a 25-year fully discrete endowment insurance, issued to (40), with sum insured 75,000.

(A) 2,188 (B) 2,190 (C) 2,192 (D) 2,194 (E) 2,196

**** END OF EXAMINATION ****

Solutions to Mock Test 1

Question #	Answer
1	B
2	E
3	E
4	D
5	C
6	A
7	E
8	D
9	B
10	C

Question #	Answer
11	A
12	A
13	B
14	A
15	C
16	C
17	D

1. [Chapter 7] Answer: (B)

   Let G be the gross premium. By the equivalence principle,

   APV of gross premiums = APV of death benefit + APV of expenses.

$$\begin{aligned} G\ddot{a}_{[x]} &= 1010\bar{A}_{[x]} + 0.9G + 0.1Ga_{[x]} + 15 + 3a_{[x]} \\ &= 1010\bar{A}_{[x]} + 0.9G + 0.1G(\ddot{a}_{[x]} - 1) + 15 + 3(\ddot{a}_{[x]} - 1) \\ G &= \frac{1010\bar{A}_{[x]} + 12 + 3\ddot{a}_{[x]}}{0.9\ddot{a}_{[x]} - 0.8} \end{aligned}$$

2. [Chapter 4] Answer: (E)

   We first change all annuities into insurances.

   Statement (ii) implies that $A_x = 1 - \frac{0.07}{1.07} \times 11.7089 = 0.2340$.

   Statement (iii) implies that $A_{x:\overline{40}|} = 1 - \frac{0.07}{1.07} \times 11.55 = 0.2444$.

   Statement (iv) implies that $A_{x+40} = 1 - \frac{0.07}{1.07} \times 7.1889 = 0.5297$.

   Finally, by $A_x = A^1_{x:\overline{40}|} + A_{x:\overset{\ \ 1}{\overline{40}|}}A_{x+40}$ and $A_{x:\overset{\ \ 1}{\overline{40}|}} + A^1_{x:\overline{40}|} = A_{x:\overline{40}|}$,

$$0.2340 = A^1_{x:\overline{40}|} + (0.2444 - A^1_{x:\overline{40}|}) \times 0.5297$$

   On solving, we get $A^1_{x:\overline{40}|} = \dfrac{0.2340 - 0.2444 \times 0.5297}{1 - 0.5297} = 0.2223$.

3. [Chapter 2] Answer: (E)

   <u>Method 1:</u>

$$\begin{aligned} {}_{1.8}p_{[66]+0.6} &= {}_{0.4}p_{[66]+0.6} \times {}_{1.4}p_{[66]+1} \\ &= {}_{0.4}p_{[66]+0.6} \times p_{[66]+1} \times {}_{0.4}p_{[66]+2} \\ &= {}_{0.4}p_{[66]+0.6} \times p_{[66]+1} \times {}_{0.4}p_{68} \end{aligned}$$

   Obviously, $p_{[66]+1} = 1 - 0.135 = 0.865,\ {}_{0.4}p_{68} = 1 - 0.4q_{68} = 1 - 0.4(0.16) = 0.936$.

   Finally, ${}_{0.4}p_{[66]+0.6} = p_{[66]}/{}_{0.6}p_{[66]} = 0.88/(1 - 0.6 \times 0.12) = 0.948276$.

   So, the answer is ${}_{1.8}p_{[66]+0.6} = 0.76776$.

   <u>Method 2:</u> ${}_{1.8}p_{[66]+0.6} = l_{[66]+2.4}/l_{[66]+0.6}$

   Without loss of generality, let $l_{[66]} = 100$. Then we have:

   $l_{[66]+1} = 88,\ l_{[66]+2} = 88 \times 0.865 = 76.12,\ l_{[66]+2} = 76.12 \times 0.84 = 63.9408$.

   So, $l_{[66]+2.4} = 0.4 \times 63.9408 + 0.6 \times 76.12 = 71.24832$,

   $l_{[66]+0.4} = 0.6 \times 88 + 0.4 \times 100 = 92.8$, and ${}_{1.8}p_{[66]+0.6} = 0.767762$.

