ACTEX Learning

Study Manual for Exam FM

2nd Edition 2nd Printing

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An SOA Exam

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QUESTION 19 OF 704 | Question # | Go! | Prev | Next

Question — Difficulty: Advanced

An airport purchases an insurance policy to offset costs associated with excessive amounts of snowfall. The insurer pays the airport 300 for every full ten inches of snow in excess of 40 inches, up to a policy maximum of 700.

The following table shows the probability function for the random variable X of annual (winter season) snowfall, in inches, at the airport.

Inches	[0,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	[90,inf)
Probability	0.06	0.18	0.26	0.22	0.14	0.06	0.04	0.04	0.00

Calculate the standard deviation of the amount paid under the policy.

Possible Answers

A 134 | ✓ 235 | X 271 | D 313 | E 352

Help Me Start

Find the probabilities for the four possible payment amounts: 0, 300, 600, and 700.

Solution

With the amount of snowfall as X and the amount paid under the policy as Y, we have

y	$f_Y(y) = P(Y = y)$
0	$P(Y = 0) = P(0 \le X < 50) = 0.72$
300	$P(Y = 300) = P(50 \le X < 60) = 0.14$
600	$P(Y = 600) = P(60 \le X < 70) = 0.06$
700	$P(Y = 700) = P(Y \ge 70) = 0.08$

The standard deviation of Y is $\sqrt{E(Y^2) - [E(Y)]^2}$.

$$E(Y) = 0.14 \times 300 + 0.06 \times 600 + 0.08 \times 700 = 134$$

$$E(Y^2) = 0.14 \times 300^2 + 0.06 \times 600^2 + 0.08 \times 700^2 = 73400$$

$$\sqrt{E(Y^2) - [E(Y)]^2} = \sqrt{73400 - 134^2} = 235.465$$

Common Questions & Errors

Students shouldn't overthink the problem with fractional payments of 300. Also, account for probabilities in which payment cap of 700 is reached.

In these problems, we must distinguish between the REALT RV (how much snow falls) and the PAYMENT RV (when does the insurer pay)?. The problem states "The insurer pays the airport 300 for every full ten inches of snow in excess of 40 inches, up to a policy maximum of 700 ." So the insurer will not start paying UNTIL AFTER 10 full inches in excess of 40 inches of snow is reached (say at 50+ or 51). In other words, the insurer will pay nothing if X<50.

Rate this problem: Excellent | Needs Improvement | Inadequate

Quickly access the Hub for additional learning.

Flag problems for review, record notes, and get instructor support.

View difficulty level.

Helpful strategies to get you started.

Full solutions with detailed explanations to deepen your understanding.

Commonly encountered errors.

Rate a problem or give feedback.

Preface

ACTEX first published a study manual for the Society of Actuaries' Exam FM ("Financial Mathematics") in 2004. That manual was prepared by lead author Matthew Hassett, assisted by Michael Ratliff, Toni Coombs Garcia, and Amy Steeby. The manual has been regularly updated and expanded to keep pace with changes in the SOA's Exam FM syllabus. This latest edition of the ACTEX Study Manual for Exam FM, edited by lead author John Dinius, has been updated to reflect the current SOA syllabus (which is effective for exams administered in October 2022 and later). It provides over 1,000 examples, exercises, and problems to help you prepare for Exam FM.

This manual contains 9 learning modules, the first 7 of which cover all the material required for Exam FM. Modules 1 and 2 provide in-depth explanations and methodologies for the basic concepts of interest theory (the Time Value of Money and Annuities). Modules 3 and 4 apply these concepts to practical situations (Loans and Bonds). Modules 5, 6, and 7 cover more advanced topics, including Internal Rate of Return, the Term Structure of Interest Rates, and Asset-Liability Management. The last two modules (8 and 9) are no longer part of the Exam FM syllabus, but their material may be of interest to students who want to expand their knowledge of interest theory.

The manual provides three "midterm exams" that allow you to test your knowledge of the material you just learned. These exams are located after Modules 2, 4, and 7. At the end of the manual there are 14 full-length practice exams of 30 problems each. All of the problems in these midterm and practice exams are original and are not available anywhere else. They are intended to provide a realistic exam-taking experience to help you complete your preparation for Exam FM.

The following pages provide recommendations on how to prepare for actuarial exams and suggestions for using this manual most effectively.

Any errors in this manual that are identified after it is published will be posted on the ACTEX website (www.actexlearning.com/errata). We suggest that you access that link (selecting ACTEX Study Materials and Exam FM) and look for "ACTEX Exam FM Study Manual, 2nd Edition, 2nd Printing." If errata for this manual have been posted, you should make the appropriate changes in your copy of the manual.

If you find a possible error in this manual, please let us know about it. You can click on the "Contact Us" link on the ACTEX homepage (www.actexlearning.com/contact-us) and describe the issue. We will review and respond to all comments. Any confirmed errata will be posted on the ACTEX website under the "Errata" link.

On Passing Exams

How to Learn Actuarial Mathematics and Pass Exams

On the next page you will find a list of study tips for learning the material in the Exam FM syllabus and passing Exam FM. But first it is important to state the basic learning philosophy that we are using in this guide:

You must master the basics before you proceed to the more difficult problems.

Think about your basic calculus course. There were some very challenging applications in which you used derivatives to solve hard max-min problems.

It is important to learn how to solve these hard problems, but if you did not have the basic skills of taking derivatives and manipulating algebraic expressions, you could not do the more advanced problems. Thus every calculus book has you practice derivative skills before presenting the tougher sections on applied problems.

You should approach interest theory the same way. The first 2 or 3 modules give you the basic tools you will need to solve the problems in the later modules. Learn these concepts and methods (and the related formulas) very well, as you will need them in each of the remaining modules.

This guide is designed to progress from simpler problems to harder ones.

In each module we start with the basic concepts and simple examples, and then progress to more difficult material so that you will be prepared to attack actual exam problems by the end of the module.

The same philosophy is used in our practice exams at the end of this manual. The first few practice exams have simpler problems, and the problems become more difficult as you progress through the practice exams.

A good strategy when taking an exam is to answer all of the easier problems before you tackle the harder ones.

An exam is scored in percentage terms, and a multiple choice exam like Exam FM will have a mix of problems at different difficulty levels.

If an exam has ten problems and three are very hard, getting the right answers to only the three hard problems and missing the others gets you a score of 30%. This could happen if the hardest problems are the first ones on the exam and you attempt them first and never get to the easy problems.

A useful exam strategy is to go through the exam and quickly solve all the more basic problems before spending extra time on the hard ones. Strive to answer all of the easy problems correctly.

Study Tips

1) Develop a schedule so that you will complete your studying in time for the exam. Divide your schedule into time for each module, plus time at the end to review and to solve practice problems. Your schedule will depend on how much time you have before the exam, but a reasonable approach might be to complete one module per week.

2) If possible, join a study group of your peers who are studying for Exam FM.

3) For each module:

    a) Read the module in the FM manual (and the associated SOA study note, if any).

    b) As you read through the examples in the text, make sure that you can correctly compute the answers.

    c) Summarize each concept you learn in the manual's margins (if you have the print version) or in a notebook.

    d) Understand the main idea of each concept and be able to summarize it in your own words. Imagine that you are trying to teach someone else this concept.

    e) While reading, create flash cards for the formulas to facilitate memorization.

    f) Learn the calculator skills thoroughly and know *all* of the calculator's functions.

    g) Do the Basic Review Problems and review your solutions.

    h) Do the Sample Exam Problems and review your solutions.

        i) If you have been stuck on a problem for more than 20 minutes, it is OK to refer to the solutions. Just make sure that when you are finished with the problem, you can recite the concept that you missed and summarize it in your own words. If you get stuck on a problem, think about what principles were used in this question and see if you could write a different problem with similar structure (as if you were the exam writer).

        ii) Mark each sample exam problem as an Easy, Medium, or Hard problem.

    i) Do the Supplemental Exercises and review your solutions.

4) After learning the material in each module, it is a good idea to go back to previous modules and do a quick half-hour or 1-hour review, so that information isn't forgotten.

5) Go back and redo the sample exam problems that you marked as Medium or Hard when you worked through them the first time.

6) At the end of Modules 2, 4, and 7, we have included practice exams that are like midterms. Taking these tests will help you consolidate your knowledge.

7) After learning the material in all of the modules and taking the midterms, go to the practice exams.

    a) The first 6 practice exams are relatively straightforward to enable you to review the basics of each topic.

    b) The next 5 practice exams introduce more difficult questions in order to replicate the actual exam experience.

    c) The last 3 practice exams include especially challenging problems to test your understanding of the material and your ability to apply solution techniques.

    Please keep in mind that the actual exam questions are confidential, and there is no guarantee that the questions you encounter on Exam FM will look exactly like the problems in this manual.

Contents

Introduction

As you begin your preparation for the Society of Actuaries' Exam FM, you should be aware that studying Financial Mathematics (or "interest theory," as I like to call it) is not a matter of learning mathematics. Instead, financial mathematics involves *applying* mathematics to situations that involve financial transactions. This will require you to learn a new language, the language of the financial world, and then to apply your *existing* math skills to solve problems that are presented in this new language. It is important that you spend adequate time to fully understand the meanings of all the terms that will be introduced in this manual. Nearly all of the problems on Exam FM will be word problems (rather than just formulas), and it is very difficult to solve these problems unless you understand the language that is being used.

In this manual, we assume that you have a solid working knowledge of differential and integral calculus and some familiarity with probability. We also assume that you have an excellent knowledge of algebraic methods. Depending on what mathematics courses you have taken (and how recently), you may need to review these topics in order to understand some of the material and work the problems in this manual.

Throughout this manual, a large number of the examples and practice problems are solved using the Texas Instruments BA II Plus calculator, which is the financial calculator approved for use on Exam FM. It is essential for you to have a BA II Plus calculator in order to understand the solutions presented here, and also to solve the problems on the actual exam. This calculator is available in a standard model, and also as the "BA II Plus *Professional*." The Professional model, which is somewhat more expensive than the standard model, is a bit easier to work with, which could be important when taking a timed exam.

At the end of the manual there is an appendix with information about the BA II Plus. This appendix is included to help you learn the calculator's functions and adjust its settings so that you will be able to solve problems more quickly. Very importantly, the appendix explains that your calculator will be "reset" by the exam staff when you check in to take Exam FM, and provides instructions for returning the calculator to the settings you prefer. Reading this material will help you avoid having calculator difficulties on the day of your exam.

Over the years, most actuarial students have found that the best way to prepare for Exam FM is to work a very large number of problems (hundreds and hundreds of problems). There are many examples, exercises, problems, and practice exams included in this manual. Many more problems can be found on the Society of Actuaries website (www.soa.org) or by searching the Web. You should plan to spend a significant proportion of your study time working problems and reviewing the solutions that are provided in this manual and on the websites.

Financial mathematics is an integral part of an actuary's skill set, and you can expect to apply interest theory regularly throughout your career. Mastering the topics covered in this manual will provide you a valuable tool for understanding financial and economic matters both on and off the job.

Best of luck to you in learning Financial Mathematics and passing Exam FM!

John Dinius
May 2024

Module 2

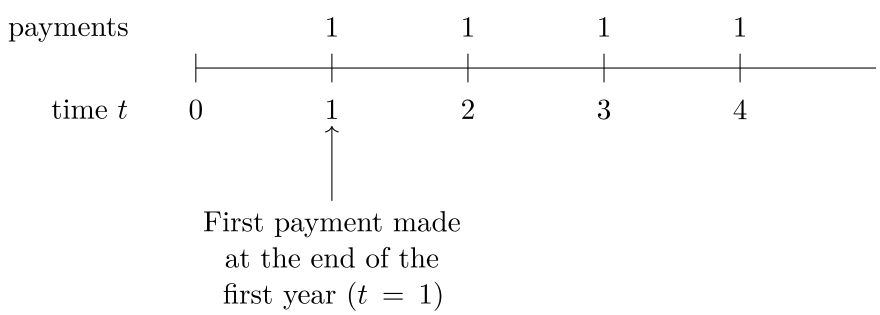
Annuities

Section 2.1 Introduction to Annuities

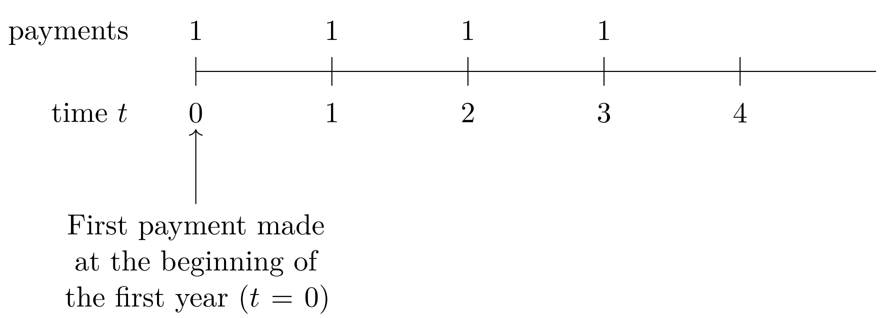
Many financial obligations require a series of periodic payments. Mortgage and car loan payments are usually made monthly. A retiree's pension plan typically pays a set amount at the beginning of every month. Premiums for an insurance policy might be paid monthly, quarterly, semi-annually, or annually. Series of regular payments such as these are called **annuities**. If the payments continue for a fixed **period** (e.g., 10 years), the annuity is called an **annuity-certain**. If the payment period is *not* fixed (e.g., a pension plan that makes monthly payments only as long as the retiree survives), it is a *contingent* annuity. (In the case of the pension, it is a "*life*-contingent annuity," or simply a "life annuity.") Exam FM deals with annuities-certain. Contingent annuities are covered in later exams.

A **unit annuity** is one for which each periodic payment is 1. Annuity payments may be made at the beginning or the end of each time period. If an annuity's payments occur at the end of each period, it is called an **annuity-immediate**. If the payments are made at the beginning of each period, it is an **annuity-due**. The diagrams below illustrate the payment patterns for unit annuities with four annual payments.

Annuity-Immediate



Annuity-Due



The preceding diagrams are called **timelines**. You will find them to be very useful in visualizing payment patterns and solving annuity problems.

Geometric Series

To find the present value or future value of an annuity, we will need to use the formula for the sum of a **geometric series**. Geometric series are very important for Exam FM. Consider the geometric series with n terms where the first term is 1 and the common ratio is r:

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r} = \frac{r^n - 1}{r - 1}, \; r \neq 1 \tag{2.1}$$

In order for this equality to be valid, n must be an integer. The geometric series is not properly defined if n is not an integer.

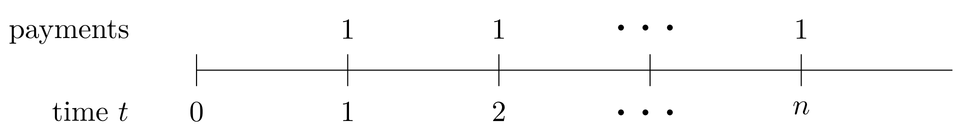
If $|r| < 1$, n can be *infinite*, because then the infinite geometric series converges (since $r^\infty = 0$):

$$1 + r + r^2 + \cdots = \frac{1}{1 - r} \quad \text{for } |r| < 1 \tag{2.2}$$

Section 2.2 Annuity-Immediate Calculations

The present value of an annuity-immediate with n annual payments of 1, calculated at an annual effective interest rate i, is denoted by $a_{\overline{n}|i}$, or we can simply write $a_{\overline{n}|}$ if the value of the interest rate is clear and does not need to be specified. When referring to $a_{\overline{n}|}$, we say "**a-angle-n**." The basic formula for $a_{\overline{n}|i}$ is so important that we will derive it here:

The present value of an n-year unit annuity-immediate is the sum of the individual present values of the n payments of 1:



As of $t = 0$, the value of the first payment is v.

As of $t = 0$, the value of the second payment is v^2.

• • •

As of $t = 0$, the value of the n^{th} payment is v^n.

$$\begin{aligned}\text{Total present value } &= a_{\overline{n}|} = v + v^2 + \cdots + v^n \\ &= v\left(1 + v + \cdots + v^{n-1}\right) \\ &= v\frac{(1-v^n)}{1-v} = v\frac{(1-v^n)}{d} \\ &= v\frac{(1-v^n)}{iv} = \frac{1-v^n}{i}\end{aligned}$$

Thus we obtain the important formula:

$$a_{\overline{n}|i} = \frac{1-v^n}{i} \tag{2.3}$$

Example (2.4)

If $i = 0.05$ and $n = 10$, then $a_{\overline{10}|5\%} = \frac{1-\left(\frac{1}{1.05}\right)^{10}}{0.05} = 7.7217$.

Calculator Note

Your calculator's **TVM worksheet** can be used to calculate the value of the annuity in Example (2.4). The PMT key is used for the periodic payment of 1. The following entries give the result PV = − 7.7217:

10	N
5	I/Y
1	PMT
0	FV
CPT	PV

Note the sign convention. Positive amounts represent money paid to you, and negative amounts represent cash that you must pay out. If the applicable interest rate is 5%, you would need to pay 7.7217 now (i.e., you would have a cash flow of −7.7217) to receive 10 subsequent payments of +1.

Note: If your calculator displays an answer of −8.1078, it is in "BGN" mode. See the Calculator Note on page 69 to learn how to correct this problem.

On exams most students use the calculator's TVM functions instead of formulas whenever possible, because it saves time. You must still know the formulas, since formula knowledge is required to solve the problems, and some questions are designed so that the calculator cannot be used directly.

Exercise (2.5)

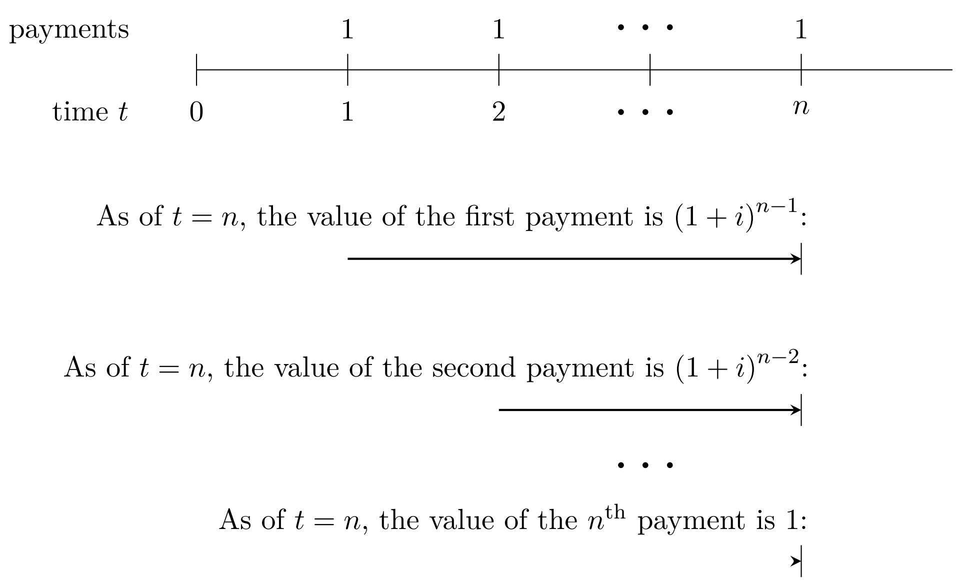
Find the value of $a_{\overline{20}|0.05}$ using the annuity formula, and then check it using your calculator's TVM functions.

Answer: 12.4622

The future value of a unit annuity-immediate with n payments is denoted by $s_{\overline{n}|}$, which is pronounced "**s-angle-n**." It is the sum of the future values (as of time n) of the n individual payments of 1.

$$s_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1$$

Note that since an annuity immediate has end-of-year payments, the first payment (made at time 1) earns interest for $n-1$ **periods** (from time 1 to time n), and the last payment of 1 (made at time n) earns no interest. It is important to understand that these are the same n payments that had a present value of $a_{\overline{n}|}$ at time 0. What is different is the *valuation date*. The present value of these payments ($a_{\overline{n}|}$) has a valuation date of time 0. The future value of these same payments ($s_{\overline{n}|}$) has a valuation date of time n.



We could find the sum of the geometric series to develop a formula for $s_{\overline{n}|}$, but we can also find the value of $s_{\overline{n}|}$ quickly based on the formula for $a_{\overline{n}|}$. Since $a_{\overline{n}|}$ is the value of this series of n payments as of time 0, and $s_{\overline{n}|}$ is the value of the same payments n periods later (at time n), we multiply $a_{\overline{n}|}$ by $(1+i)^n$ to find $s_{\overline{n}|}$:

(2.6)
$$s_{\overline{n}|i} = (1+i)^n \cdot a_{\overline{n}|i} = (1+i)^n \cdot \frac{1-v^n}{i} = \frac{(1+i)^n - 1}{i}$$

You can use this approach to avoid excessive memorization. If you know the formula for $a_{\overline{n}|}$, you can easily write the formula for $s_{\overline{n}|}$.

Example (2.7)

If $i = 5\%$ and $n = 10$,

$$s_{\overline{10}|5\%} = 1.05^{10} \cdot a_{\overline{10}|5\%} = 1.05^{10} \times 7.7217 = 12.5779, \text{ or:}$$

$$s_{\overline{10}|5\%} = \frac{1.05^{10} - 1}{0.05} = 12.5779$$

This can also be done on the financial calculator. Set N = 10, I/Y = 5, PMT = 1, and CPT FV = −12.5779. Naturally, PMT and FV have opposite signs.

Exercise (2.8)

Based on an annual effective interest rate of 6%, find $a_{\overline{15}|}$ and $s_{\overline{15}|}$.

Answers: $a_{\overline{15}|} = 9.712$ $s_{\overline{15}|} = 23.276$

To get another very useful relationship, divide both sides of Formula (**2.6**) by $(1+i)^n$:

(2.9) $$a_{\overline{n}|} = v^n \cdot s_{\overline{n}|}$$

The relationships between $a_{\overline{n}|}$ and $s_{\overline{n}|}$ in (**2.6**) and (**2.9**) are intuitive. They represent the value of n payments at time 0 and at time n, respectively, so their values differ by a factor of $(1+i)^n$, or by $(1+i)^{-n} = v^n$.

Annuities with Level Payments Other Than 1

Note that the present value or future value of *any* annuity-immediate with level payments can be found using $a_{\overline{n}|}$ and $s_{\overline{n}|}$. If an annuity-immediate has payments of amount P, its present value and future value are given by:

$$PV = P \cdot a_{\overline{n}|}$$
$$FV = P \cdot s_{\overline{n}|}$$

Example (2.10)

Find the present value of an annuity-immediate with 10 annual payments of 100, based on an annual effective interest rate of 5%:

$$100 \cdot a_{\overline{10}|} = 100 \cdot \frac{1 - \left(\frac{1}{1.05}\right)^{10}}{0.05} = 100 \times 7.7217 = 772.17$$

Exercise (2.11)

Find the present value of an annuity-immediate with 30 annual payments of 500, based on an annual effective interest rate of 8%.

Answer: 5,628.89

Section 2.3 Perpetuities

A **perpetuity** is an annuity with payments that continue forever. The present value of a perpetuity-immediate that pays 1 per period is denoted by $a_{\overline{\infty}|}$.

(2.12)
$$a_{\overline{\infty}|} = v + v^2 + v^3 + \cdots$$

If we write $a_{\overline{\infty}|}$ as a limit, we obtain the following formula:

(2.13)
$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i}$$

Example (2.14)

If $i = 5\%$, then $a_{\overline{\infty}|} = \frac{1}{0.05} = 20$.

Exercise (2.15)

Find the present value of a unit perpetuity-immediate with $i = 8\%$.

Answer: 12.50

You can think of a perpetuity as a "savings account" that pays out the interest earned each year but never pays out any of the principal. Using that concept, we can develop Formula (**2.13**) using simple algebra. Let X be the value of a perpetuity-immediate. At $t = 0$, the "savings account" has a balance of X. At $t = 1$, the balance is $X \cdot (1 + i) = X + X \cdot i$, and the interest $(X \cdot i)$ is paid out, leaving a balance of X in the account. Each year, immediately after the interest payment of $(X \cdot i)$, the balance will again be X. (This is logical, because there are always an infinite number of future payments, and X was defined as the value of the perpetuity, i.e., the value of an infinite number of future payments.) If the perpetuity's annual payment amount is 1, then we have:

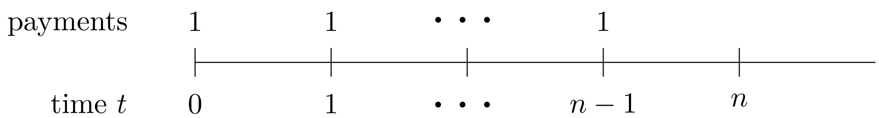
$$X \cdot i = 1 \longrightarrow X = 1/i$$

Thus X, the value of our unit perpetuity-immediate, has the same value we found for $a_{\overline{\infty}|}$ in Formula (**2.13**).

Note: We cannot write a formula for the future value of a perpetuity (which would be written $s_{\overline{\infty}|}$). That would be the perpetuity's value as of the date of its last payment, and there is no "last payment" under a perpetuity.

Section 2.4 Annuity-Due Calculations

The present value of an n-period unit **annuity-due** is denoted by $\ddot{a}_{\overline{n}|}$, which is pronounced "**a-double-dot-angle-n**."



As of $t=0$, the value of the first payment is 1.

As of $t=0$, the value of the second payment is v.

• • •

As of $t=0$, the value of the n^{th} payment (which occurs at $t=n-1$) is v^{n-1}.

Summing the present values of these payments, we have:

$$\ddot{a}_{\overline{n}|} = 1 + v + \cdots + v^{n-1} = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$$

Thus:

(2.16)
$$\boxed{\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d}}$$

Another way to develop this formula is to recognize that each of the n payments in this annuity due occurs *one period earlier* than the corresponding payment under an n year annuity immediate. As a result, $\ddot{a}_{\overline{n}|}$ has a value that is larger than $a_{\overline{n}|}$ by a factor of $(1+i)$:

(2.17)
$$\boxed{\ddot{a}_{\overline{n}|} = (1+i)\cdot a_{\overline{n}|} = (1+i)\cdot\frac{1-v^n}{i} = \frac{1-v^n}{i/(1+i)} = \frac{1-v^n}{d}}$$

The formula for $\ddot{a}_{\overline{n}|}$ is easy to remember, since it is obtained by taking the equation for $a_{\overline{n}|}$ and replacing the i in the denominator by d. As a memory aid, you might use the fact that the words "immediate" and "due" begin with the letters "i" and "d," and that i and d are their respective denominators. This pattern of denominators also applies to formulas for the *future* value of an annuity-due and the present value of a perpetuity-due:

(2.18)
$$\boxed{\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}}$$

(2.19)
$$\boxed{\ddot{a}_{\overline{\infty}|} = \frac{1}{d}}$$

We can develop some useful relationships between annuities-immediate and annuities-due by general reasoning. For example, an n-period annuity-due consists of the same payments as an $(n-1)$-period annuity-immediate plus a payment of 1 at time 0, so we can write:

(2.20) $$\boxed{\ddot{a}_{\overline{n}|} = a_{\overline{n-1}|} + 1}$$

For a perpetuity, this becomes:

(2.21) $$\boxed{\ddot{a}_{\overline{\infty}|} = a_{\overline{\infty}|} + 1}$$

The future value of an n-period annuity-immediate is equal to the future value of an $(n-1)$-period annuity-due plus an n^{th} payment at time n (the valuation date):

(2.22) $$\boxed{s_{\overline{n}|} = \ddot{s}_{\overline{n-1}|} + 1}$$

Example (2.23)

Given $i = 5\%$ and $n = 10$, find $\ddot{a}_{\overline{n}|}$ directly, and check your answer using the relationship between $\ddot{a}_{\overline{n}|}$ and $a_{\overline{n}|}$.

Solution.

$$\begin{aligned}\ddot{a}_{\overline{10}|} &= \frac{1-v^n}{d} = \frac{1-\left(\frac{1}{1.05}\right)^{10}}{\left(\frac{0.05}{1.05}\right)} \\ &= 8.1078\end{aligned}$$

Check:
From Example (2.4), $a_{\overline{10}|} = 7.7217$:

$$\begin{aligned}\ddot{a}_{\overline{10}|} &= (1+i)\cdot a_{\overline{10}|} = 1.05(7.7217) \\ &= 8.1078\end{aligned}$$

Formula (2.20) provides another way to find $\ddot{a}_{\overline{10}|}$:

$$\begin{aligned}\ddot{a}_{\overline{10}|} &= a_{\overline{9}|} + 1 = \frac{1-1.05^{-9}}{0.05} + 1 = 7.1078 + 1 \\ &= 8.1078\end{aligned}$$

Exercise (2.24)

Calculate $\ddot{a}_{\overline{15}|}$ and $\ddot{s}_{\overline{15}|}$ based on an annual effective interest rate of 6%.

Answers: $\ddot{a}_{\overline{15}|6\%} = 10.295$ $\ddot{s}_{\overline{15}|6\%} = 24.673$

Calculator Note

Annuity-due calculations are done with the calculator set to the BGN (begin) mode to reflect that payments are made at the *beginning* of the period. The letters BGN appear above the [PMT] key. If you key in [2nd] [BGN] you will see either BGN or END in the calculator's display. You can then change to the other mode by pressing [2nd] and [SET] (the 2$^{\text{nd}}$ function of the [ENTER] key). *Remember that you can leave the* BGN/END *menu by pressing the* [CE/C] *key or by pressing* [2nd] [QUIT].

You can tell whether your calculator is set to BGN or END mode by looking at the upper right of the screen. If "BGN" appears in small letters at the upper right, the calculator is in BGN (annuity-due) mode. If BGN does not appear on the screen, it is in END (annuity-immediate) mode.

It is very important on actuarial exams to be aware of your calculator's BGN/END *mode. The majority of problems require* END *mode. If you do a* BGN *mode problem and do not reset the calculator to* END *mode, you will have trouble on subsequent problems. Many students avoid this difficulty by keeping their calculators set to* END *at all times. When they need to find the value of an annuity-due, they calculate the value of the corresponding annuity-immediate, and then multiply by* $(1 + i)$.

Example (2.25)

Calculate $\ddot{a}_{\overline{10}|5\%}$ using the BA II Plus's TVM worksheet.

Solution.
Set the calculator to BGN mode. Set N = 10, I/Y = 5, PMT = −1, and FV = 0.
CPT PV = 8.1078.

We could have entered the same values with the calculator in END mode. In that case, we would CPT PV = 7.7217, the value of an annuity-immediate. Then multiply by 1.05 to get the answer: $\ddot{a}_{\overline{10}|} = 8.1078$.

Exercise (2.26)

Calculate $\ddot{a}_{\overline{15}|6\%}$ using the calculator's TVM worksheet.

Answer: 10.295

Reminder: Be sure to return your calculator to END mode.

Section 2.5 Continuously Payable Annuities

A continuous unit annuity pays a total of 1 per year, but spreads the payment evenly throughout the year by making a continuous stream of payments at a constant rate of 1 per year. We can think of this **continuously payable annuity** as making a payment of $(1 \cdot dt)$ in each infinitesimal time interval of length dt. The present value (at time 0) of a payment of $(1 \cdot dt)$ made at time t is $\left(v^t \cdot dt\right)$.

An n-year continuous unit annuity, which makes continuous payments at a rate of 1 per year from time 0 to time n, is denoted by $\bar{a}_{\overline{n}|}$ ("**a-bar-angle-n**"). Its present value is found by integrating the present value of its rate of payment $\left(v^t \cdot dt\right)$ from time 0 to time n:

$$\bar{a}_{\overline{n}|} = \int_{t=0}^{n} v^t \cdot dt = \left.\frac{v^t}{\ln(v)}\right|_{t=0}^{n}$$

$$= \frac{v^n - v^0}{\ln\left(\frac{1}{1+i}\right)} = \frac{v^n - 1}{-\delta} = \frac{1 - v^n}{\delta}$$

Note: In this derivation, we used the relation

$$\ln(v) = \ln\left(\frac{1}{1+i}\right) = -\ln(1+i) = -\delta.$$

The final result is:

(2.27) $$\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta} = \frac{i}{\delta} \cdot a_{\overline{n}|}$$

Note that this formula follows the pattern we observed in the formula for $\ddot{a}_{\overline{n}|}$. We can find $\bar{a}_{\overline{n}|}$ by changing the denominator of $a_{\overline{n}|}$ from i to δ (just as we changed it from i to d for $\ddot{a}_{\overline{n}|}$). This is equivalent to multiplying $a_{\overline{n}|}$ by $\frac{i}{\delta}$.

Similarly, for the *future* value of a continuously payable unit annuity, we have:

(2.28) $$\bar{s}_{\overline{n}|} = (1+i)^n \cdot \bar{a}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta} = \frac{i}{\delta} \cdot s_{\overline{n}|}$$

For a continuously payable unit *perpetuity*:

(2.29) $$\bar{a}_{\overline{\infty}|} = \frac{1}{\delta} = \frac{i}{\delta} \cdot a_{\overline{\infty}|}$$

Example (2.30)

For $i = 5\%$, calculate $\bar{a}_{\overline{10}|}$ and $\bar{s}_{\overline{10}|}$.

Solution.

$$\begin{aligned}\bar{a}_{\overline{10}|} &= \frac{1 - v^{10}}{\delta} = \frac{1 - \left(\frac{1}{1.05}\right)^{10}}{\ln(1.05)} \\ &= 7.9132\end{aligned}$$

This can be checked by using the $\dfrac{i}{\delta}$ adjustment factor and the fact that $a_{\overline{10}|} = 7.7217$:

$$\begin{aligned}\bar{a}_{\overline{10}|} &= \frac{i}{\delta} \cdot a_{\overline{10}|} = \frac{0.05}{\ln(1.05)}(7.7217) \\ &= 7.9132\end{aligned}$$

For $\bar{s}_{\overline{10}|}$, we have:

$$\bar{s}_{\overline{10}|} = \frac{1.05^{10} - 1}{\ln(1.05)} = 12.8898$$

We can check this value by accumulating $\bar{a}_{\overline{10}|}$ for 10 years (i.e., moving the valuation date from $t = 0$ to $t = 10$):

$$\begin{aligned}\bar{s}_{\overline{10}|} &= 1.05^{10} \cdot \bar{a}_{\overline{10}|} = 1.6289 \times 7.9132 \\ &= 12.8898\end{aligned}$$

Calculator Note

The BA II Plus has 10 memories (in addition to the memories associated with the TVM functions and the various worksheets). These 10 memories are numbered 0 to 9. To store the currently displayed value in (for example) Memory 2, press [STO] 2. Then to recall that value for use in a calculation, press [RCL] 2.

When solving a problem, you will frequently calculate an intermediate result that will be needed later in solving that problem. It is best to use your calculator's memories to store such values. It may also be useful to write down a 3- or 4-digit *approximation* of the intermediate result as a record of your work. But re-entering that *approximate* value into the calculator in place of the *calculated* value wastes time and loses accuracy. Instead, *use the calculator's memories* to maintain the *full precision* of these values.

Exercise (2.31)

For $i = 6\%$, calculate $\bar{a}_{\overline{15}|}$ and $\bar{s}_{\overline{15}|}$.

Answers: $\bar{a}_{\overline{15}|} = 10.0008$ $\bar{s}_{\overline{15}|} = 23.9675$

Example (2.32)

A 20-year continuous stream of payments consists of payments at a rate of 3,000 per year for the first 10 years, then at a rate of 2,000 per year from $t = 10$ to $t = 20$. At an interest rate of 6% convertible monthly, what is the present value of this payment stream?

Solution.
The payment stream can be broken into two parts: continuous payments at a rate of 2,000 per year for the full 20-year period, and continuous payments of 1,000 per year for just the first 10 years. The total present value is:

$$2,000 \cdot \bar{a}_{\overline{20}|} + 1,000 \cdot \bar{a}_{\overline{10}|} = 2,000 \cdot \frac{1 - v^{20}}{\delta} + 1,000 \cdot \frac{1 - v^{10}}{\delta}$$

In order to evaluate this expression, we need values for v and δ:

$$\begin{aligned} v &= \frac{1}{1+i} = \frac{1}{\left(1 + \frac{i^{(12)}}{12}\right)^{12}} = \frac{1}{\left(1 + \frac{0.06}{12}\right)^{12}} = \frac{1}{1.06168} \\ &= 0.9419 \end{aligned}$$

$$\begin{aligned} \delta &= \ln(1+i) = \ln(1.06168) \\ &= 0.05985 \end{aligned}$$

Using these values, the present value of this payment stream is:

$$\begin{aligned} 2,000 \cdot \frac{1 - v^{20}}{\delta} + 1,000 \cdot \frac{1 - v^{10}}{\delta} &= 2,000 \cdot \frac{1 - 0.9419^{20}}{0.05985} + 1,000 \cdot \frac{1 - 0.9419^{10}}{0.05985} \\ &= 2,000 \times 11.6607 + 1,000 \times 7.5249 \\ &= 30,846.44 \end{aligned}$$

Exercise (2.33)

An account pays interest at a continuously compounded rate of 0.05 per year. Continuous deposits are made to the account at a rate of 1,000 per year for 6 years, and then at a rate of 2,000 per year for the next 4 years. What is the account balance at the end of 10 years?

Answer: 17,402.48

Section 2.6 Basic Annuity Problems for Calculator Practice

We can solve for each of the variables PMT, PV, FV, I/Y, and N using the BA II Plus. In this section we give an example of solving for each of these using the BA II Plus calculator. In each case, a formula is also given, indicating how the problem can be solved using annuity functions.

Example (2.34)

Solving for PMT
A loan for 20,000 is to be repaid by 5 year-end payments with interest at an annual effective rate of 12%. What is the amount of the annual payment?

Solution.
The 5 payments must have a present value equal to the amount of the loan:

$$P \cdot a_{\overline{5}|12\%} = 20,000$$

Set N = 5, I/Y = 12, and PV = 20,000. CPT PMT = −5,548.19.

The annual payment is 5,548.19.

Exercise (2.35)

A loan for 15,000 is to be repaid by 8 year-end payments with interest at an annual effective rate of 10%. What is the amount of the annual payment?

Answer: 2,811.66

Example (2.36)

Solving for PMT
You have a 5,000 balance in an account earning a 4.5% annual effective rate. You want to increase your balance to 20,000 at the end of 12 years by making a level deposit of D at the beginning of each of the next 12 years. Find D, the required level annual deposit.

Solution.

$$5,000 \times 1.045^{12} + D \cdot \ddot{s}_{\overline{12}|4.5\%} = 20,000$$

Put the calculator in BGN mode. Set N = 12, I/Y = 4.5, PV = −5,000, and FV = 20,000. CPT PMT = −712.91.

The level annual deposit is 712.91.

The problem of Example (**2.36**) could also have been solved with the calculator in END mode. In that case, you would enter the same values, but would need to adjust the result:

Set N = 12, I/Y = 4.5, PV = −5,000, and FV = 20,000. CPT PMT = −744.99.

744.99 is the amount you would need to deposit at the *end* of each year. Since this problem involves deposits made one year earlier (at the *beginning* of each year), the deposits should be smaller by a factor of $1/(1+i)$: $\frac{744.99}{1.045} = 712.91$.

Exercise (2.37)

What is the required level deposit in (**2.36**) if the current balance is 4,000 and the annual effective interest rate is 6%?

Answer: 668.33

If you have changed your calculator setting to BGN mode, be sure to reset it to END mode.

Example (2.38)

Solving for PV

You wish to make a deposit now to an account earning a 5% annual effective rate so that you can withdraw 1,000 at the end of each of the next 15 years. How much should you deposit today?

Solution.

$$D = 1,000 \cdot a_{\overline{15}|5\%}$$

Set N = 15, I/Y = 5, and PMT = 1,000. CPT PV = −10,379.66.

You should deposit 10,379.66.

Exercise (2.39)

What would be the required deposit in Example (**2.38**) if you wanted 20 years of withdrawals instead of 15?

Answer: 12,462.21

Example (2.40)

Solving for FV

An account earning a 5% annual effective rate has a current balance of 6,000. If a deposit of 1,500 is made at the end of each year for 20 years, what will be the balance in the account at the end of 20 years?

Solution.

$$Bal_{20} = 6,000 \cdot 1.05^{20} + 1,500 \cdot s_{\overline{20}|5\%}$$

Set N = 20, I/Y = 5, PV = −6,000, and PMT = −1,500. CPT FV = 65,518.72.

The balance at the end of 20 years will be 65,518.72.

Exercise (2.41)

What would be the ending balance in Example (**2.38**) if the interest rate were 6% instead of 5%?

Answer: 74,421.20

Example (2.42)

Solving for I/Y

You have borrowed 15,000 and agreed to repay the loan with 5 level annual payments of 4,000, with the first payment occurring one year from the date of the loan. What annual effective interest rate are you paying?

Solution.

$$15,000 = 4,000 \cdot a_{\overline{5}|i}$$

Set N = 5, PV = 15,000, and PMT = −4,000. CPT I/Y = 10.42.

You are paying interest at a 10.42% annual effective rate.

Note that PV is positive, since it represents cash you received; PMT *is negative, because it is cash that you must pay. If you forget the minus sign, the* BA II Plus *will give an error message when you press* [CPT] [I/Y].

Exercise (2.43)

What would the interest rate be in Example (**2.42**) if the annual loan payment were 4,300?

Answer: 13.34%

Note: The equations of value in Example (**2.42**) *and Exercise* (**2.43**) *involve* 5^{th} *degree polynomials in* v *or* $(1+i)$. *There is no formula to solve for* i *directly. Instead, the TVM worksheet "iterates" (uses successive approximations) until it finds a value for* i *that satisfies the equation within a small margin of error.*

Example (2.44)

Solving for N

You want to accumulate at least 20,000 in an account earning a 5% annual effective rate. You will make a level deposit of 1,000 at the beginning of each year for n years. What is the value of n? What is the account balance after n years?

Solution.

The equation of value is $20,000 = 1,000 \cdot \ddot{s}_{\overline{n}|5\%}$.

We can find n using algebra, as follows:

$$20,000 = 1,000 \cdot \ddot{s}_{\overline{n}|5\%} = 1,000 \cdot \frac{1.05^n - 1}{0.05/1.05}$$

$$20 \cdot (0.05/1.05) + 1 = 1.05^n$$

$$n = \frac{\ln[20 \cdot (0.05/1.05) + 1]}{\ln 1.05} = 13.71$$

The *computed* value for n is 13.71. This answer means that 13 payments are not enough, and a 14th payment is required to reach 20,000. Thus $n = 14$.

$$1,000 \cdot \ddot{s}_{\overline{14}|5\%} = 1,000 \cdot \frac{1.05^{14} - 1}{0.05/1.05} = 20,578.56$$

The account balance after 14 years is 20,578.56.
To solve by calculator, put the BA II Plus into BGN mode and set I/Y = 5, PMT = −1,000, and FV = 20,000. CPT N = 13.71.

Now enter 14 for N and CPT FV = 20,578.56.

(Or, to solve the problem in END mode, make the same entries, except enter −1,050 for PMT ($1,050 = 1,000 \times 1.05$) to reflect that a payment of 1,050 at year-end is equivalent to a payment of 1,000 at the beginning of the year.)

A note about the answer to Example (2.44):

It is important to understand that the annuity formula $a_{\overline{n}|} = \frac{1-v^n}{i}$ is valid only if n is an integer, because it is based on the geometric series formula, which requires n to be an integer. The calculated value of 13.71 in Example (**2.44**) does not mean that the account balance is 20,000 at time 13.71 years. Rather, it simply tells us that the balance will be less than 20,000 at time 13 (after 13 payments), and it will be more than 20,000 at time 14 (after 14 payments).

We can calculate the amount of the *partial* payment needed at the beginning of the 14th year to produce a balance of *exactly* 20,000 at the end of 14 years. We have already calculated that the balance at time 14 will be 20,578.56. This suggests that we could have deposited $\frac{578.56}{1.05} = 551.01$ less at the beginning of the 14th year (at time 13). Thus a final payment of 448.99 ($= 1,000 - 551.01$) at time 13 would produce a balance of exactly 20,000 at time 14.

Alternatively, if the deposit at the beginning of the 14th year is the full 1,000, we can calculate when during the 14th year the balance will reach 20,000. After the deposit at time 13, the balance is $1,000 \cdot \ddot{s}_{\overline{13}|5\%} + 1,000 = 19,598.63$. Let p be the fraction of a year required for this amount to grow to 20,000. Then:

$$19,598.63 \cdot 1.05^p = 20,000$$

$$1.05^p = 20,000/19,598.63 = 1.02048$$

$$p = \frac{\ln 1.02048}{\ln 1.05} = 0.4155$$

The balance will reach 20,000 after 13.4155 years.

Or, using the **BA II Plus**, set I/Y = 5, PV = −19,598.63, PMT = 0, and FV = 20,000.

CPT N = 0.4155.

In this case, the non-integer value of N is valid. Because there are no periodic payments (PMT = 0), there is no geometric series, so N does not have to be an integer.

Exercise (2.45)

How many payments (n) would be needed in Example (**2.44**) if the interest rate were 6%? What payment amount at the beginning of the n^{th} year would produce a balance of exactly 20,000 at time n?

Answers: 13 985.79

Section 2.7 Annuities with Varying Payments

Not all series of payments are level. In practice, it is quite common to encounter **annuities with varying payments**, such as the following:

Series of Payments:	***Payments Made:***	***Type of Annuity Sequence:***
500, 0, 200, 300	At end of period	Irregular
1, 2, 3, 4	At end of period	Arithmetic increasing
4, 3, 2, 1	At end of period	Arithmetic decreasing
1, 1.05, $1.1025 = 1.05^2$	Beginning of period	Geometric

In interest theory, there are formulas for the last three sequences presented here, and they will be covered in the following sections. But if there are only four or five terms to input, you can calculate the annuity's value more quickly by using your calculator's **Cash Flow worksheet**.

For example, if $i = 0.05$, you can find the present value of the increasing annuity-immediate $\{1, 2, 3, 4\}$ by using the [CF] (Cash Flow) and [NPV] (Net Present Value) keys:

First press the [CF] key to activate the Cash Flow worksheet. Then press [2nd] [CLR WORK] to clear any cash flow values that were previously entered. You will see a prompt for the value of CF_0, the cash flow at time 0. In this case, there is no payment until time 1, so leave the CF_0 value at 0 and press the [↓] key. You will see "C01," requesting the cash flow at time 1. Press 1 and [ENTER]. Scroll down again, and there will be a new prompt: "F01." This is a request for the number of times (frequency) this value is repeated. The default value is 1, and if you scroll down again, the value of 1 will be assumed with no entry. After you scroll down, you will be prompted for the value of C02. Press 2 and [ENTER]. Repeat this process until all four cash flow values have been entered. Then calculate the NPV at 5% with the following keystrokes:

[NPV] [5] [ENTER] [↓] [CPT]

The display will show the answer: NPV = 8.6488

To find the present value of the first series $(500, 0, 200, 300)$ at $i = 0.05$, enter the cash flows and use the NPV function with I = 5. Result: NPV = 895.77.

Note $CF_0 = 0$, because CF_0 is the payment at $t = 0$. The first payment is at the end of the first period ($t = 1$). Thus, $CF_0 = 0$, C01=500, C02=0, C03=200, and C04=300.

To calculate the present value of the last series in the table, enter 1 for CF_0. (Payments are made at the beginning of each period, so the first payment is at $t = 0$.) Then enter 1.05 for C01, and 1.1025 for C02. At a 6% annual effective interest rate, the present value is 2.9718.

Section 2.8 Increasing Annuities with Terms in Arithmetic Progression

An annuity whose n payments are 1, 2, 3, …, n is called a **unit increasing annuity**. If payments are made at the *end* of each period, it is an increasing annuity-immediate. The present value of this annuity is denoted by $(Ia)_{\overline{n|}}$ and, of course, $(Ia)_{\overline{n|}}$ is equal to the present value of the annuity's payments:

$$(Ia)_{\overline{n|}} = v + 2v^2 + 3v^3 + \ldots + n \cdot v^n.$$

To develop a practical formula for this function, first note that the payment pattern is as follows:

payments		1	2	3	• • •	n
time t	0	1	2	3	• • •	n

These same payments can be arranged as shown in the following table. This arrangement allows us to write a formula for the present value of the payments in each line. We can then sum those present values to create a formula for the value of $(Ia)_{\overline{n|}}$:

time t	1	2	3	⋯	n	Present Value at $t = 0$
payments	1	1	1	⋯	1	$\frac{1-v^n}{i}$
		1	1	⋯	1	$v \cdot \frac{1-v^{n-1}}{i} = \frac{v-v^n}{i}$
			1	⋯	1	$v^2 \cdot \frac{1-v^{n-2}}{i} = \frac{v^2-v^n}{i}$
					…	…
					1	$v^{n-1} \cdot \frac{1-v}{i} = \frac{v^{n-1}-v^n}{i}$
Total	1	2	3		n	$\frac{\sum_{t=0}^{n-1} v^t - n \cdot v^n}{i} = \frac{\ddot{a}_{\overline{n|}} - n \cdot v^n}{i}$

The result is:

(2.46)
$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n \cdot v^n}{i}$$

Example (2.47)

Let $i = 5\%$ and $n = 4$. Then the increasing annuity payments are 1, 2, 3, 4, and the present value is:

$$(Ia)_{\overline{4|}} = \frac{\ddot{a}_{\overline{4|}} - 4 \cdot v^4}{0.05} = 8.6488$$

Note that 8.6488 is the same value we calculated for this sequence of payments in the previous section using the Cash Flow worksheet.

Exercise (2.48)

Find $(Ia)_{\overline{15|}i=6\%}$.

Answer: 67.2668

As with level annuities, to create a formula for an increasing unit annuity-*due* or a continuously payable increasing annuity, we just change the denominator:

(2.49) $$(I\ddot{a})_{\overline{n|}} = \frac{i}{d} \cdot (Ia)_{\overline{n|}} = (1+i) \cdot (Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n \cdot v^n}{d}$$

(2.50) $$(I\bar{a})_{\overline{n|}} = \frac{i}{\delta} \cdot (Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - n \cdot v^n}{\delta}$$

In the case of an increasing perpetuity, the present value is the limit of the appropriate formula as n approaches infinity:

(2.51) $$(Ia)_{\overline{\infty|}} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n|}} - n \cdot v^n}{i} = \frac{\frac{1}{d} - 0}{i} = \frac{1}{id}$$

(2.52) $$(I\ddot{a})_{\overline{\infty|}} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n|}} - n \cdot v^n}{d} = \frac{\frac{1}{d} - 0}{d} = \frac{1}{d^2}$$

(2.53) $$(I\bar{a})_{\overline{\infty|}} = \lim_{n\to\infty} \frac{\ddot{a}_{\overline{n|}} - n \cdot v^n}{\delta} = \frac{\frac{1}{d} - 0}{\delta} = \frac{1}{\delta d}$$

Note: In Formulas (**2.50**) *and* (**2.53**), *payments are made continuously, but the increases occur annually. That is, continuous payments are made at a rate of 1 per year during the* 1^{st} *year, 2 per year during the* 2^{nd} *year, etc.*

Example (2.54)

Let $i = 5\%$ and $n = 4$.

Then, $(I\ddot{a})_{\overline{4}|5\%} = 1.05 \cdot (Ia)_{\overline{4}|5\%} = 1.05 \times 8.6488 = 9.0812$.

Note: 8.6488 is the value calculated for $(Ia)_{\overline{4}|5\%}$ in Example (2.47).

Exercise (2.55)

Find $(I\ddot{a})_{\overline{15}|6\%}$.

Answer: 71.3028

The *future* value of an increasing unit annuity-immediate is denoted by $(Is)_{\overline{n}|}$. One can avoid excessive memorization of formulas by using the relationship $(Is)_{\overline{n}|} = (1+i)^n \cdot (Ia)_{\overline{n}|}$. (The value as of the date of the n^{th} payment equals the value as of time 0, $(Ia)_{\overline{n}|}$, accumulated n periods to time n.) Below, we show the commonly used expressions for calculating $(Is)_{\overline{n}|}$, $(I\ddot{s})_{\overline{n}|}$, and $(I\bar{s})_{\overline{n}|}$.

(2.56)
$$(Is)_{\overline{n}|} = (1+i)^n \cdot (Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

(2.57)
$$(I\ddot{s})_{\overline{n}|} = (1+i)^n \cdot (I\ddot{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{i}{d} \cdot (Is)_{\overline{n}|}$$

(2.58)
$$(I\bar{s})_{\overline{n}|} = (1+i)^n \cdot (I\bar{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{\delta} = \frac{i}{\delta} \cdot (Is)_{\overline{n}|}$$

!

The number of formulas here appears overwhelming, but the situation is relatively simple, and the adjustment factors are the same as for level annuities.

If you can calculate $(Ia)_{\overline{n}|}$, then to obtain the value of any other type of increasing annuity, simply do a multiplication:

Multiply by: $(1+i)^n$ to convert $(Ia)_{\overline{n}|}$ to $(Is)_{\overline{n}|}$.

Multiply by i/d to convert to an annuity-due.

Multiply by i/δ to convert to a continuously-payable annuity.

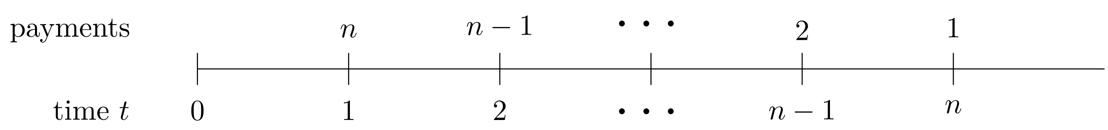
Extensive memorization is not required!

Section 2.9 Decreasing Annuities with Terms in Arithmetic Progression

The n-year **unit decreasing annuity-immediate** has n payments:

$$n, n-1, \ldots, 1, \text{ payable at times } 1, 2, 3, \ldots, n$$

Its present value is denoted by $(Da)_{\overline{n}|}$.



To develop a formula for $(Da)_{\overline{n}|}$, first note that $(Da)_{\overline{n}|} + (Ia)_{\overline{n}|} = (n+1) \cdot a_{\overline{n}|}$. (At time 1, $(Da)_{\overline{n}|}$ pays n and $(Ia)_{\overline{n}|}$ pays 1, for a total of $n+1$; at time 2, $(Da)_{\overline{n}|}$ pays $n-1$ and $(Ia)_{\overline{n}|}$ pays 2, for a total of $n+1$, etc. So in each of the n years, the total payment is $n+1$.)

We can rearrange that equation and solve for $(Da)_{\overline{n}|}$:

$$(Da)_{\overline{n}|} = (n+1) \cdot a_{\overline{n}|} - (Ia)_{\overline{n}|} = (n+1) \cdot \frac{1-v^n}{i} - \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{i}$$

$$= \frac{(n+1) - \ddot{a}_{\overline{n}|} - v^n}{i} = \frac{n - \left(\ddot{a}_{\overline{n}|} - 1 + v^n\right)}{i} = \frac{n - a_{\overline{n}|}}{i}$$

Note: The last step in this derivation uses the relationship $\ddot{a}_{\overline{n}|} - 1 + v^n = a_{\overline{n}|}$. (If we start with an annuity-due, $\ddot{a}_{\overline{n}|}$, then subtract the payment at time 0 (worth 1) and add a payment at time n (worth v^n), the result is an annuity-immediate, $a_{\overline{n}|}$.)

As with arithmetic increasing annuities, you really need only one formula for arithmetic decreasing annuities:

(2.59) $$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

There are also formulas for the present value of a decreasing annuity-*due* and a decreasing continuously-payable annuity, as well as for the *future* values of these annuities. But, as shown here, their values can also be found by simple adjustments to the formula for $(Da)_{\overline{n}|}$:

(2.60) $$(D\ddot{a})_{\overline{n}|} = (1+i) \cdot (Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d}$$

(2.61) $$(D\bar{a})_{\overline{n}|} = \frac{i}{\delta} \cdot (Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{\delta}$$

(2.62) $$(Ds)_{\overline{n}|} = (1+i)^n \cdot (Da)_{\overline{n}|} = \frac{n \cdot (1+i)^n - s_{\overline{n}|}}{i}$$

(2.63)
$$(D\ddot{s})_{\overline{n}|} = (1+i)^n \cdot (D\ddot{a})_{\overline{n}|} = \frac{n \cdot (1+i)^n - s_{\overline{n}|}}{d}$$

(2.64)
$$(D\bar{s})_{\overline{n}|} = (1+i)^n \cdot (D\bar{a})_{\overline{n}|} = \frac{n \cdot (1+i)^n - s_{\overline{n}|}}{\delta}$$

Example (2.65)

Given $i = 5\%$, find $(Da)_{\overline{4}|}$.

Solution.

$$(Da)_{\overline{4}|} = \frac{4 - a_{\overline{4}|}}{0.05} = \frac{4 - 3.546}{0.05} = 9.08$$

This can be checked on the BA II Plus using the CF workbook and applying the NPV function to the sequence 4, 3, 2, 1 with I=5. (CF0=0, C01=4, C02=3, etc.)

Exercise (2.66)

Find $(Da)_{\overline{15}|}$ for $i = 0.06$.

Answer: 88.1292

Example (2.67)

Find $(D\bar{s})_{\overline{10}|}$ at an annual effective interest rate of 8%.

Solution.

$$(D\bar{s})_{\overline{10}|} = \frac{10 \times 1.08^{10} - s_{\overline{10}|8\%}}{\delta} = \frac{21.5892 - 14.4866}{\ln 1.08} = \frac{7.1027}{0.07696}$$
$$= 92.29$$

Exercise (2.68)

Find $(D\ddot{s})_{\overline{12}|}$ for $i = 0.05$.

Answer: 118.2961

Section 2.10 A Single Formula for Annuities with Terms in Arithmetic Progression

Suppose the first payment in an **arithmetic annuity-immediate** is P and the subsequent payments change by Q per period, where Q can be either positive or negative. If the annuity has n payments, the sequence of payments is:

$$P, P+Q, P+2Q, \ldots, P+(n-1)\,Q$$

The present value of this annuity is:

(2.69)
$$PV = P \cdot a_{\overline{n}|} + Q \cdot \left(\frac{a_{\overline{n}|} - n \cdot v^n}{i} \right)$$

To understand this formula, we will analyze it in two steps:

Because there is a payment of P at the end of each year for n years, the present value of the P component is $P \cdot a_{\overline{n}|}$.

The Q-related payments in years 1 through n are 0, Q, $2Q$, $3Q$, ..., $(n-1) \cdot Q$.

This is an $(n-1)$-year arithmetic increasing annuity, with its first payment at t=2. The present value of these payments is:

$$Q \cdot v \cdot (Ia)_{\overline{n-1}|} = Q \cdot v \cdot \frac{\ddot{a}_{\overline{n-1}|} - (n-1) \cdot v^{n-1}}{i} = Q \cdot \left(\frac{a_{\overline{n-1}|} - (n-1) \cdot v^n}{i} \right) = Q \cdot \left(\frac{a_{\overline{n}|} - n \cdot v^n}{i} \right)$$

Note: The last step in this development uses the relationship $a_{\overline{n-1}|} + v^n = a_{\overline{n}|}$. (An $(n-1)$-year annuity-immediate, plus a payment at time n, equals an n-year annuity-immediate.)

The total of the P and Q components combined is $P \cdot a_{\overline{n}|} + Q \cdot \left(\frac{a_{\overline{n}|} - n \cdot v^n}{i} \right)$, as in Formula (**2.69**), above. The coefficient of Q in this formula, $\left(\frac{a_{\overline{n}|} - n \cdot v^n}{i} \right)$, is very similar to Formula (**2.46**) for $(Ia)_{\overline{n}|}$, but it uses $a_{\overline{n}|}$ instead of $\ddot{a}_{\overline{n}|}$. If you know the formula for $(Ia)_{\overline{n}|}$, it is relatively easy to learn the ***PQ* formula**.

Note also that $(Ia)_{\overline{n}|}$ is the special case where $P = 1$ and $Q = 1$. Similarly, $(Da)_{\overline{n}|}$ is the special case where $P = n$ and $Q = -1$. The PQ formula thus provides an alternative way to calculate values for $(Ia)_{\overline{n}|}$ and $(Da)_{\overline{n}|}$.

We can multiply Formula (**2.69**) by $(1+i)^n$ to develop a formula for the *future* value of the same series of payments as of time n:

(2.70)
$$FV = P \cdot s_{\overline{n}|} + Q \cdot \left(\frac{s_{\overline{n}|} - n}{i} \right)$$

Note that the limit of (**2.69**) as n approaches infinity gives the present value of an increasing perpetuity-immediate of the form P, $P+Q$, $P+2Q$, . . .:

(2.71) For a perpetuity-immediate: $PV = \dfrac{P}{i} + \dfrac{Q}{i^2}$

This formula has been used in past exams. For example, see Sample Exam Problem #19 at the end of this module.

In the case of an annuity-due or perpetuity-due with payments that follow the PQ pattern, the PV and FV can be found by multiplying the above formulas by $1+i$ (or by i/d):

Annuity-due: $$PV = P \cdot \ddot{a}_{\overline{n}|} + Q \cdot \left(\frac{a_{\overline{n}|} - n \cdot v^n}{d} \right)$$

$$FV = P \cdot \ddot{s}_{\overline{n}|} + Q \cdot \left(\frac{s_{\overline{n}|} - n}{d} \right)$$

Perpetuity-due: $$PV = \frac{P}{d} + \frac{Q}{id}$$

Some students prefer to solve increasing and decreasing annuity problems using *only* the PQ formula, because it can be applied to *any* arithmetic annuity. We recommend that you learn the PQ formula *in addition to* (Ia) and (Da), since each formula has time-saving advantages in some types of problems.

Example (2.72)

A 10-year annuity-immediate has a first-year payment of 500. The subsequent payments increase by 100 each year. Find the present value of this annuity at an annual effective interest rate of 5%.

Solution.

$$PV = P \cdot a_{\overline{n}|} + Q \cdot \left(\frac{a_{\overline{n}|} - n \cdot v^n}{i} \right), \text{ where } n = 10, P = 500, \text{ and } Q = 100$$

$$PV = 500 \cdot \frac{1 - 1.05^{-10}}{0.05} + 100 \cdot \left(\frac{\frac{1-1.05^{-10}}{0.05} - \frac{10}{1.05^{10}}}{0.05} \right) = 7,026.07$$

Exercise (2.73)

A 5-year annuity-immediate has a first-year payment of 1,000. The subsequent payments decrease by 100 each year. Find the present value of this annuity at an annual effective interest rate of 6%.

Answer: 3,418.91

Calculator Note

The BA II Plus calculator does not have special functions or worksheets to calculate values for arithmetic annuities. However, there are ways to use the Time Value of Money keys to save time and effort in these calculations. First, consider the PQ formula for the present value of an arithmetic annuity. The terms of this formula can be rearranged as follows:

$$PV = P \cdot a_{\overline{n}|} + Q \cdot \left(\frac{a_{\overline{n}|} - n \cdot v^n}{i} \right) = \left(P + \frac{Q}{i} \right) \cdot a_{\overline{n}|} - \frac{n \cdot Q}{i} \cdot \; v^n$$

Now we can use the TVM worksheet to calculate PV.

Set N $= n$, I/Y $= i$, PMT $= \left(P + \frac{Q}{i} \right)$, and FV $= -\frac{n \cdot Q}{i}$.

Then CPT PV (and change its sign) to find the value of PV.

Applying this method to the facts of Example (**2.72**), set N $= 10$, I/Y $= 5$, PMT $= 500 + 100/.05 = 2,500$, FV $= -\frac{10 \cdot 100}{.05} = -20,000$. CPT PV $= -7{,}026.07$.

The present value of this stream of payments is 7,026.07.

To find the present value of an increasing annuity, $(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - n \cdot v^n}{i}$, calculate the value of the numerator by setting N $= n$, I/Y $= i$, PMT $= 1$, and FV $= -n$. Set the calculator to BGN mode, press CPT PV, and change the sign. The result is the numerator. Divide by i to calculate $(Ia)_{\overline{n}|}$.

There is a trick you can use to avoid putting your calculator into BGN mode. Since $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n}|} - v^n$, we can write $(Ia)_{\overline{n}|} = \frac{1 + a_{\overline{n}|} - (n+1) \cdot v^n}{i}$. Now we can set FV $= -(n+1)$ instead of $-n$, CPT PV, change its sign, and add 1 to get the value of the numerator. Then divide the result by i to find the value of $(Ia)_{\overline{n}|}$.

As an example of this latter procedure, the value of the increasing annuity of Exercise (**2.48**) can be calculated as follows:

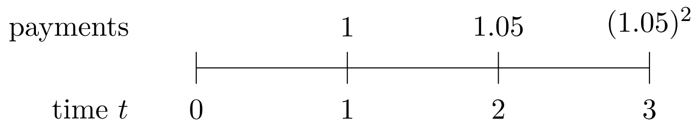
Set N $= 15$, I/Y $= 6$, PMT $= 1$, and FV $= -16$. Then CPT PV $= -3.0360$.

Now change the sign of PV and add 1 to get the numerator: 4.0360.

Finally, divide by i: $4.0360/.06 = 67.2668$.

Section 2.11 Annuities with Terms in Geometric Progression

Consider the sequence of three payments 1, 1.05, and $(1.05)^2$ made at the end of years 1, 2, and 3:



These payments increase geometrically, with a common ratio of 1.05. This is a **geometric annuity** with a growth rate $g = 0.05$.

Suppose that we wish to find the present value of this geometric annuity at an annual effective interest rate of $i = 10\%$. Applying the formula for the sum of a geometric series, we have:

$$PV = \frac{1}{1.10} + \frac{1.05}{(1.10)^2} + \frac{(1.05)^2}{(1.10)^3} = \frac{1}{1.10}\underbrace{\left[1 + \frac{1.05}{1.10} + \left(\frac{1.05}{1.10}\right)^2\right]}_{\substack{\text{This is a geometric series} \\ \text{with } n=3 \text{ and } r=\frac{1.05}{1.10}.}}$$

$$= \frac{1}{1.10}\left[\frac{1-\left(\frac{1.05}{1.10}\right)^3}{1-\left(\frac{1.05}{1.10}\right)}\right] = 2.6052$$

More generally, we can consider an n-year geometric annuity-immediate with growth rate g. Its payments are $1, (1+g), (1+g)^2, \ldots, (1+g)^{n-1}$, and its present value is represented by the symbol $a^g_{\overline{n}|i}$ or $a^g_{\overline{n}|}$. (*Note: This actuarial symbol for geometric annuities is not widely used, but it is descriptive and will be used in this manual.*) The first payment in the series represented by $a^g_{\overline{n}|}$ is 1, and the annual rate of change, g, can be either positive or negative, reflecting payments that increase or decrease geometrically.

There are three standard methods for calculating the value of a geometric annuity. We will present all three in this module. You may choose to learn only one of the three, or you might prefer to learn all three, so that you can apply the one that is best suited to a particular problem.

Geometric Series Method

Using the same approach as above, we can express the present value of a geometric annuity-immediate as follows:

$$a^g_{\overline{n}|i} = \frac{1}{(1+i)} + \frac{(1+g)}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \cdots + \frac{(1+g)^{n-1}}{(1+i)^n}$$

$$= \frac{1}{1+i}\left[1 + \left(\frac{1+g}{1+i}\right) + \left(\frac{1+g}{1+i}\right)^2 + \cdots + \left(\frac{1+g}{1+i}\right)^{n-1}\right]$$

The quantity in brackets is a geometric series with common ratio $r = \frac{(1+g)}{(1+i)}$, so we can apply the formula for the sum of a geometric series to find its value:

$$a^{g}_{\overline{n}|i} = \left(\frac{1}{1+i}\right) \cdot \frac{1-\left(\frac{1+g}{1+i}\right)^n}{1-\left(\frac{1+g}{1+i}\right)} \tag{2.74}$$

In a similar manner, the geometric series formula can be applied to find the value of any geometric annuity (including present and future values of geometric annuities-immediate and annuities-due). Of course, g and i must each be constant throughout the term of the annuity.

Note that if $g = i$, the formula cannot be applied (because the denominator is 0). But in that case all of the terms of the geometric series are equal, so the sum equals n times the value of the first term:

$$\begin{aligned} \text{If } g = i: \quad a^{g}_{\overline{n}|i} &= \frac{1}{1+i}\left[1 + \left(\frac{1+g}{1+i}\right) + \left(\frac{1+g}{1+i}\right)^2 + \cdots + \left(\frac{1+g}{1+i}\right)^{n-1}\right] \\ &= \frac{1}{1+i}[1+1+1+\cdots+1] = \frac{n}{1+i} \end{aligned} \tag{2.75}$$

In the case of a perpetuity (where n is infinite), the present value can be determined, provided that $\left|\frac{1+g}{1+i}\right| < 1$, because then we have $\lim_{n\to\infty} \left(\frac{1+g}{1+i}\right)^n = 0$, so the numerator of the fraction in Formula (**2.74**) becomes simply 1:

$$a^{g}_{\overline{\infty}|i} = \left(\frac{1}{1+i}\right) \cdot \frac{1}{1-\left(\frac{1+g}{1+i}\right)} \qquad \left|\frac{1+g}{1+i}\right| < 1 \tag{2.76}$$

Geometric Annuity Formula Method

It is also possible to develop *standard* formulas for the present value and future value of a geometric annuity-immediate or annuity-due. For the present value of an annuity-immediate, we can simplify Formula (**2.74**) to produce the following standard formula:

$$a^{g}_{\overline{n}|i} = \left(\frac{1}{1+i}\right) \cdot \frac{1-\left(\frac{1+g}{1+i}\right)^n}{1-\left(\frac{1+g}{1+i}\right)} = \frac{1-\left(\frac{1+g}{1+i}\right)^n}{i-g} \tag{2.77}$$

Note that when g is 0, the above formula for a geometric annuity simplifies to $(1-v^n)/i$, which is the standard formula for $a_{\overline{n}|}$, the present value of a level annuity.
The perpetuity version of Formula (**2.77**) is easy to remember. If i is the interest rate and g is the rate of growth, the present value is simply:

$$a^{g}_{\overline{\infty}|} = \frac{1}{i-g} \qquad \left|\frac{1+g}{1+i}\right| < 1 \tag{2.78}$$

Note: $\left|\frac{1+g}{1+i}\right|$ *must be less than 1; otherwise, the present value of the perpetuity is infinite.*

To modify Formula (**2.77**) for geometric annuities-due or for the future value of a geometric annuity, you can simply multiply Formula (**2.77**) by the appropriate accumulation factor to produce the following formulas:

(2.79)
$$\ddot{a}^{g}_{\overline{n}|} = (1+i)\cdot a^{g}_{\overline{n}|} = \frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{\frac{i-g}{1+i}} = \frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{d-g\cdot v}$$

(2.80)
$$s^{g}_{\overline{n}|} = (1+i)^{n}\cdot a^{g}_{\overline{n}|} = \frac{(1+i)^{n}-(1+g)^{n}}{i-g}$$

(2.81)
$$\ddot{s}^{g}_{\overline{n}|} = (1+i)^{n+1}\cdot a^{g}_{\overline{n}|} = \frac{(1+i)^{n}-(1+g)^{n}}{\frac{i-g}{1+i}} = \frac{(1+i)^{n}-(1+g)^{n}}{d-g\cdot v}$$

For a perpetuity-due, we have:

(2.82)
$$\ddot{a}^{g}_{\overline{\infty}|} = \frac{1}{d-g\cdot v} = \frac{1+i}{i-g} \qquad \left|\frac{1+g}{1+i}\right| < 1$$

If you choose to use the Geometric Annuity Formula method, we recommend that you memorize Formula (**2.77**) for the present value of a geometric annuity immediate, and modify it as necessary for other situations (as shown in the above formulas), rather than memorizing all of these formulas.

Artificial Interest Rate Method

We will again begin with the present value formula we developed from the geometric series, and will modify it to find the present value of a geometric annuity-due:

$$a^{g}_{\overline{n}|i} = \left(\frac{1}{1+i}\right)\cdot\frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{1-\left(\frac{1+g}{1+i}\right)} \qquad \ddot{a}^{g}_{\overline{n}|i} = (1+i)\cdot a^{g}_{\overline{n}|i} = \frac{1-\left(\frac{1+g}{1+i}\right)^{n}}{1-\left(\frac{1+g}{1+i}\right)}$$

Now define an "artificial interest rate" j, such that $1+j=\frac{1+i}{1+g}$.

The formula for a geometric annuity-due then becomes:

(2.83)
$$\ddot{a}^{g}_{\overline{n}|i} = \frac{1-(1+j)^{-n}}{1-(1+j)^{-1}} = \frac{1-v_j^{n}}{d_j} = \ddot{a}_{\overline{n}|j} \qquad \text{where } 1+j=\frac{1+i}{1+g}$$

Note that v_j and d_j are used to represent the present value factor and rate of discount based on the artificial interest rate j.

In a way, this is the simplest of the 3 methods, because it uses the familiar formula for an annuity-due, but with an interest rate j that has been calculated from $1+j=\frac{1+i}{1+g}$. However, it is important to remember that *this formula applies only to the present value of an annuity-due*. For an annuity-*immediate* or for *future* values, you must first calculate the present value of the annuity-due, and then adjust it by the appropriate interest factor, using $(1+i)$, *not* $(1+j)$:

$$a^{g}_{\overline{n}|i}=\frac{\ddot{a}_{\overline{n}|j}}{1+i} \qquad s^{g}_{\overline{n}|i}=(1+i)^{n-1}\cdot\ddot{a}_{\overline{n}|j} \qquad \ddot{s}^{g}_{\overline{n}|i}=(1+i)^{n}\cdot\ddot{a}_{\overline{n}|j}$$

Example (2.84)

Given $i=10\%$, find the present value of the sequence of payments:

$$1.05,(1.05)^2,\ldots,(1.05)^{10}$$

Payments are made at the beginning of each year.

Solution.
Note that this series starts with 1.05, not 1, and that payments are made at the beginning of each period.

By geometric series:

$$\begin{aligned}1.05\cdot\ddot{a}^{0.05}_{\overline{10}|0.10}&=1.05\cdot\left[1+\frac{1.05}{1.10}+\left(\frac{1.05}{1.10}\right)^2+\cdots+\left(\frac{1.05}{1.10}\right)^9\right]=1.05\cdot\frac{1-\left(\frac{1.05}{1.10}\right)^{10}}{1-\left(\frac{1.05}{1.10}\right)}\\&=8.59\end{aligned}$$

By the geometric annuity formula:

$$1.05\cdot\ddot{a}^{0.05}_{\overline{10}|0.10}=1.05\cdot\frac{1-\left(\frac{1.05}{1.10}\right)^{10}}{d-g\cdot v}=1.05\cdot\frac{1-\left(\frac{1.05}{1.10}\right)^{10}}{\frac{0.10}{1.10}-\frac{0.05}{1.10}}=8.59$$

By the artificial interest rate method:

$$1+j=\frac{1+i}{1+g}=\frac{1.10}{1.05}=1.047619$$

$$j=4.7619\%$$

$$1.05\cdot\ddot{a}^{0.05}_{\overline{10}|0.10}=1.05\cdot\ddot{a}_{\overline{10}|0.047619}=1.05\cdot\frac{1-1.047619^{-10}}{0.047619/1.047619}=8.59$$

Exercise (2.85)

Given $i=8\%$, find the present value of the following sequence of payments:

$$1,1.06,(1.06)^2,\ldots,(1.06)^9$$

Payments are made at the end of each year.

Answer: 8.5246

Example (2.86)

Given $i = 10\%$, find the present value of a perpetuity with annual payments of:

$$1.05, (1.05)^2, \ldots, (1.05)^n, \ldots.$$

a) if payments are made at the end of the year
b) if payments are made at the beginning of the year

Solution.
By **geometric series**:

a) $1.05 \cdot a^{0.05}_{\overline{\infty}|0.10} = 1.05 \cdot \frac{1}{1.10} \cdot \left[1 + \frac{1.05}{1.10} + \left(\frac{1.05}{1.10}\right)^2 + \cdots\right] = \frac{1.05}{1.10} \cdot \frac{1}{1-\frac{1.05}{1.10}} = 21$

b) $1.05 \cdot \ddot{a}^{0.05}_{\overline{\infty}|0.10} = (1+i) \cdot 1.05 \cdot a^{0.05}_{\overline{\infty}|0.10} = 1.10 \times 21 = 23.1$

By the **geometric annuity formula**:

a) $1.05 \cdot a^{0.05}_{\overline{\infty}|0.10} = 1.05 \cdot \frac{1}{i-g} = \frac{1.05}{0.10-0.05} = 21$

b) $1.05 \cdot \ddot{a}^{0.05}_{\overline{\infty}|0.10} = 1.05 \cdot \frac{1}{d-v \cdot g} = 1.05 \cdot \frac{1}{\frac{0.10}{1.10} - \frac{0.05}{1.10}} = 23.1$

By the **artificial interest rate** method, doing part b) first:

b) $1 + j = \frac{1+i}{1+g} = \frac{1.10}{1.05} = 1.047619 \qquad j = 4.7619\%$

$1.05 \cdot \ddot{a}^{0.05}_{\overline{\infty}|0.10} = 1.05 \cdot \ddot{a}_{\overline{\infty}|4.7619\%} = 1.05 \cdot \frac{1}{0.047619/1.047619} = 23.1$

a) $1.05 \cdot a^{0.05}_{\overline{\infty}|0.10} = \frac{1.05 \cdot \ddot{a}^{0.05}_{\overline{\infty}|0.10}}{1+i} = \frac{23.1}{1.10} = 21$ (Divide by $1+i$, not $1+j$.)

Exercise (2.87)

Given $i = 8\%$, find the present value of a perpetuity with payments:

$$1.075, (1.075)^2, \ldots, (1.075)^n, \ldots.$$

a) if payments are made at the end of the year
b) if payments are made at the beginning of the year

Answers: a) 215 b) 232.20

Example (2.88)

You want to save 1,000,000 for retirement. You plan to make annual deposits at the beginning of each year into an account that earns an annual effective rate of 7.5%. You will increase the amount of your deposit each year by 4%. If you plan to retire in 40 years, what should be the amount of your first deposit?

Solution.
Let D be the amount of the first deposit.

By **geometric series**:

$$\begin{aligned}1,000,000 =& D\cdot\left(1.075^{40}+1.04\cdot 1.075^{39}+1.04^2\cdot 1.075^{38}+\ldots+1.04^{39}\cdot 1.075\right)\\ =& D\cdot 1.075^{40}\cdot\left[1+\frac{1.04}{1.075}+\left(\frac{1.04}{1.075}\right)^2+\cdots+\left(\frac{1.04}{1.075}\right)^{39}\right]\\ =& D\cdot 1.075^{40}\cdot\frac{1-\left(\frac{1.04}{1.075}\right)^{40}}{1-\left(\frac{1.04}{1.075}\right)}=406.756\cdot D\\ D =& \frac{1,000,000}{406.756}=2,458.48\end{aligned}$$

By the **geometric annuity formula**:

$$\begin{aligned}1,000,000 &= D\cdot \ddot{s}^{0.04}_{\overline{40}|7.5\%}=D\cdot\frac{(1+i)^{40}-(1+g)^{40}}{d-g\cdot v}\\ &= D\cdot\frac{1.075^{40}-1.04^{40}}{\frac{0.075}{1.075}-\frac{0.04}{1.075}}=406.756\cdot D\\ D &= \frac{1,000,000}{406.756}=2,458.48\end{aligned}$$

By the **artificial interest rate** method:

$$1+j=\frac{1+i}{1+g}=\frac{1.075}{1.04}=1.033654 \qquad j=3.3654\%$$

$$\begin{aligned}1,000,000 &= D\cdot \ddot{s}^{0.04}_{\overline{40}|7.5\%}=D\cdot \ddot{a}^{0.04}_{\overline{40}|7.5\%}\cdot 1.075^{40}=D\cdot \ddot{a}_{\overline{40}|j}\cdot 1.075^{40}\\ &= D\cdot\frac{1-1.033654^{-40}}{\frac{0.033654}{1.033654}}\cdot 1.075^{40}=406.756\cdot D\\ D &= \frac{1,000,000}{406.756}=2,458.48\end{aligned}$$

Exercise (2.89)

If the deposits in Example (**2.88**) increase by 5% each year, what is the amount of the first deposit?

Answer: 2,113.35

Section 2.12 Equations of Value and Loan Payments

We have already used equations of value to solve annuity problems. In **Section 2.6** we solved problems that involved finding the periodic payment for a loan or determining the amount (or the number) of level deposits needed to accumulate a targeted amount. We also solved those problems using the BA II Plus's TVM worksheet. Now we will discuss in more detail the use of equations of value to solve annuity problems.

Suppose that you borrow 10,000 at an interest rate $i = 8\%$ and will make level loan payments at the end of each year for 10 years. How do you find the annual payment P?

The principle that is used to find P is that the present value of the borrower's loan payments must equal the present value of the amount borrowed.

In this case, we have: $10,000 = P \cdot a_{\overline{10}|8\%} \longrightarrow P = \dfrac{10,000}{a_{\overline{10}|8\%}} = \dfrac{10,000}{6.7101} = 1,490.29$

Our valuation date for this equation is $t = 0$, the date of the loan. We could have chosen a different valuation date (such as $t = 10$), but $t = 0$ is convenient because we already know the value of the amount borrowed as of that date (10,000).

> **!** The computation is simple, but the key point here is the principle involved: the two sets of payments (those made by the lender, and those made by the borrower) must have *the same value as of the valuation date*. In other words, the value received by the borrower must equal the value repaid to the lender.

Note that the loan payment amount could easily be calculated on a financial calculator:
Set N = 10, I/Y = 8, PV =10,000, and CPT PMT.

In order to understand the calculator's TVM calculations, it is useful to know that the calculator uses the following equation of value (assuming it is in **END** mode):

$$\text{PV} + \text{PMT} \cdot \frac{1 - \left(1 + \frac{\text{I/Y}}{100}\right)^{-\text{N}}}{\frac{\text{I/Y}}{100}} + \text{FV} \cdot \left(1 + \frac{\text{I/Y}}{100}\right)^{-\text{N}} = 0$$

In actuarial notation, this would be:

$$\text{PV} + \text{PMT} \cdot a_{\overline{n}|} + \text{FV} \cdot (1 + i)^{-n} = 0$$

Since the present values of PV, PMT, and FV are all added together (on the left side of this equation), it is clear that at least one must be positive and at least one must be negative in order for the total to be 0. For example, if you lend money, then you are giving that money to the borrower (entered as negative), and you will receive one or more payments from the borrower (entered as positive). The underlying formulas for this and other calculator functions can be found in the Appendix of this manual.

The following example demonstrates that the valuation date for an equation of value does not have to be the date of the first or last cash flow.

Example (2.90)

You will make 3 deposits of 10,000 each into a bank account, at the beginning of this year and the following 2 years. You then plan to withdraw a level payment P starting at the beginning of year 4 and continuing for five years. The account pays interest at an annual effective rate $i = 8\%$. What is the amount of the level payment P?

Solution.
The following timeline illustrates the cash flows in this problem.

Deposits	10,000	10,000	10,000					
time t	0	1	2	3	4	5	6	7
Withdrawals				P	P	P	P	P

Note that the first withdrawal occurs at the beginning of year 4, which is $t = 3$.
(Year 4 begins at $t = 3$ and ends at $t = 4$.)

We will determine the values of the deposits and the withdrawals as of time $t = 3$. In other words, our valuation date is $t = 3$.

$$[\text{value of deposits as of } t = 3] = [\text{value of withdrawals as of } t = 3],$$

$$10,000 \cdot \ddot{s}_{\overline{3}|} = P \cdot \ddot{a}_{\overline{5}|}$$

$$P = 10,000 \cdot \frac{\ddot{s}_{\overline{3}|}}{\ddot{a}_{\overline{5}|}}$$

$$= 10,000 \cdot \frac{3.5061}{4.3121}$$

$$= 8,130.82$$

Note: We could choose a different valuation date, such as $t = 0$, but the formulas are simpler with $t = 3$. ($t = 2$ would also work well.)

Exercise (2.91)

If the withdrawals in Example (**2.90**) begin at the end of the 4$^{\text{th}}$ year and continue for ten years, what is the amount of each withdrawal?

Answer: 5,225.14

Section 2.13 Deferred Annuities

There are cases in which you may want to find the present value of an annuity with payments that begin in some future period. For example, you might plan to retire in 5 years and would like to purchase an annuity now that will pay you 10,000 at the end of each year for 10 years, starting 5 years from now. The present value of this annuity would be $v^5 \cdot 10{,}000 a_{\overline{10}|}$. (Note that in this example the first payment will occur at the end of the 6th year.)

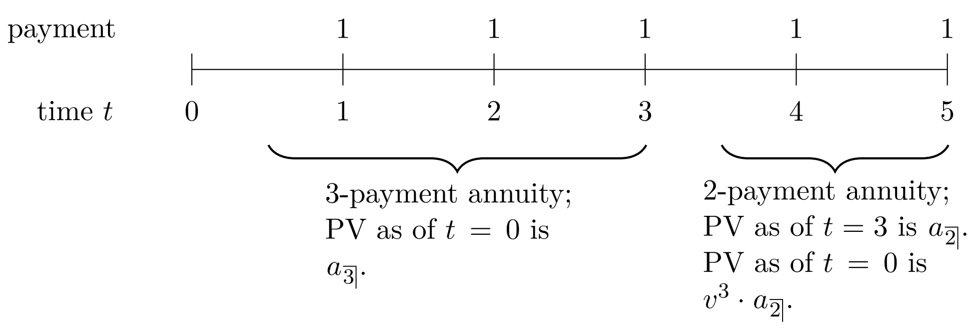
An annuity such as this is called a **deferred annuity**. In general, the present value of an ***n*-year unit annuity-immediate deferred for *k* years** is $v^k \cdot a_{\overline{n}|}$. One form of notation for such an annuity is ${}_{k|}a_{\overline{n}|}$, so we have:

(2.92)
$$ {}_{k|}a_{\overline{n}|} = v^k \cdot a_{\overline{n}|} $$

There is an identity that breaks down the present value of an annuity-immediate into the sum of two shorter annuities, one that begins now and one that is deferred. We will illustrate this with an example.

The present value of a 5-year annuity immediate can be expressed as the sum of a 3-year annuity-immediate plus a 2-year annuity-immediate that is deferred for 3 years:

$$ a_{\overline{5}|} = v + v^2 + v^3 + v^4 + v^5 = \left(v + v^2 + v^3\right) + v^3\left(v + v^2\right) = a_{\overline{3}|} + v^3 \cdot a_{\overline{2}|} = a_{\overline{3}|} + {}_{3|}a_{\overline{2}|} $$



Total PV at $t = 0$: $\quad a_{\overline{3}|} + v^3 \cdot a_{\overline{2}|}$

Thus the present value of a 5-period unit annuity immediate can be broken down into the present value of a 3-period annuity and the present value of a 2-period annuity that begins in 3 periods.

This reasoning works in general, so we have:

$$ a_{\overline{n+k}|} = a_{\overline{k}|} + {}_{k|}a_{\overline{n}|} = a_{\overline{k}|} + v^k \cdot a_{\overline{n}|} $$

This shows that the present value of an annuity-immediate for $n + k$ periods is the sum of the present value of a k-period annuity-immediate and an n-period annuity deferred for k periods.

We can rewrite this identity as a formula for the value of a deferred annuity:

(2.93)
$$ {}_{k|}a_{\overline{n}|} = a_{\overline{n+k}|} - a_{\overline{k}|} $$

Example (2.94)

Based on a 5% annual effective interest rate, find the present value of a 10-year annuity with level annual payments of 100, where the first payment will occur 4 years from now.

Solution.
This is a 3-year-deferred annuity-immediate (no payments during the first 3 years; first payment at the end of the 4$^{\text{th}}$ year).

$$\begin{aligned} 100 \cdot {}_{3|}a_{\overline{10}|} &= 100 \cdot v^3 \cdot \frac{1-v^{10}}{i} \\ &= 100 \times 1.05^{-3} \cdot \frac{1-1.05^{-10}}{0.05} \\ &= 667.03 \end{aligned}$$

Alternatively,

$$\begin{aligned} 100 \cdot \left(a_{\overline{n+k}|} - a_{\overline{k}|}\right) &= 100 \cdot \left(a_{\overline{13}|} - a_{\overline{3}|}\right) \\ &= 100 \cdot (9.3936 - 2.7232) \\ &= 667.03 \end{aligned}$$

Exercise (2.95)

Based on a 5% annual effective interest rate, find the present value of a 10-year annuity with level annual payments of 100 each, with the first payment occurring at the beginning of the 6$^{\text{th}}$ year.

Answer: 635.27

If we apply Formula (**2.93**) to perpetuities (i.e., set $n = \infty$), we have:

$${}_{k|}a_{\overline{\infty}|} = a_{\overline{\infty}|} - a_{\overline{k}|}$$

By rearranging this equation, we can produce the standard formula for the present value of a k-period annuity:

$$a_{\overline{k}|} = a_{\overline{\infty}|} - {}_{k|}a_{\overline{\infty}|} = \frac{1}{i} - v^k \cdot \frac{1}{i} = \frac{1-v^k}{i} \tag{2.96}$$

This formula shows that a k-year annuity is equivalent to a perpetuity, less a k-year-deferred perpetuity.

Section 2.14 Annuities With More Complex Payment Patterns

We pointed out previously that the BA II Plus's CF worksheet and NPV function can be used to evaluate non-level annuities. In some problems involving **non-level payments**, we can instead apply the calculator's TVM functions in two (or more) steps, as we see in the next problem.

Example (2.97)

An annuity pays 1 at the end of each of the next four years and 2 at the end of each of the four following years. Based on a 5% annual effective interest rate, what is the present value of this annuity?

Solution.
This sequence of payments can be broken down into a 4-year annuity-immediate with payments of 1, plus a 4-year-deferred annuity-immediate with 4 payments of 2 each:

$$\begin{aligned} a_{\overline{4}|} + 2 \cdot {}_{4|}a_{\overline{4}|} &= \frac{1-1.05^{-4}}{0.05} + 2 \times 1.05^{-4} \times \frac{1-1.05^{-4}}{0.05} \\ &= 9.38 \end{aligned}$$

This can be done on the calculator by calculating $a_{\overline{4}|}$ (set N = 4, I/Y = 5, PMT = −1, and CPT PV), and then multiplying $a_{\overline{4}|}$ by $\left(1 + 2\,(1.05)^{-4}\right)$.

Here are 2 other approaches:

- An 8 year annuity-immediate with payments of 1, plus a 4-year-deferred 4-year annuity-immediate with payments of 1:

$$\begin{aligned} a_{\overline{8}|} + {}_{4|}a_{\overline{4}|} &= \frac{1-1.05^{-8}}{0.05} + 1.05^{-4} \times \frac{1-1.05^{-4}}{0.05} \\ &= 9.38 \end{aligned}$$

- An 8-year annuity-immediate with payments of 2, MINUS a 4-year annuity-immediate with payments of 1:

$$\begin{aligned} 2 \cdot a_{\overline{8}|} - a_{\overline{4}|} &= 2 \cdot \frac{1-1.05^{-8}}{0.05} - \frac{1-1.05^{-4}}{0.05} \\ &= 9.38 \end{aligned}$$

Exercise (2.98)

An annuity pays 100 at the end of each of the next 10 years and 200 at the end of each of the five subsequent years. If $i = 0.08$, find the present value of the annuity.

Answer: 1,040.89

An annuity's payments can vary in many different patterns, as you will see when you look at the Sample Exam Problems at the end of this module. The next two examples illustrate this.

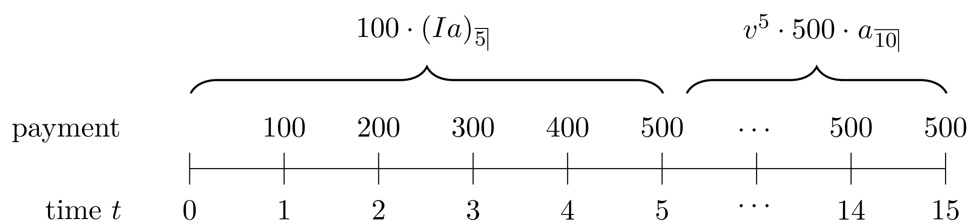
Example (2.99)

An annuity-immediate has a first payment of 100, and the payments increase by 100 each year until they reach 500. There are 10 further payments of 500. Find the present value of this annuity at 6.5%.

Solution.
The equation of value is

$$\begin{aligned} PV &= 100 \cdot (Ia)_{\overline{5}|} + v^5 \cdot 500 \cdot a_{\overline{10}|} \\ &= 100 \times 11.945 + 0.7299 \times 500 \times 7.189 \\ &= 3,817.95 \end{aligned}$$

The following diagram shows how the payments were grouped to develop this formula.



As in the previous example, there is more than one way to break this annuity's payments into components that we can value. In this example, we found a way to express the present value using $(Ia)_{\overline{5}|}$ and ${}_{5|}a_{\overline{10}|}$. Since the 5^{th}-year payment of 500 can be either the 5^{th} payment of the increasing annuity or the first payment of the level annuity, we could also have calculated the answer as:

$$\begin{aligned} PV &= 100 \cdot (Ia)_{\overline{4}|} + v^4 \cdot 500 \cdot a_{\overline{11}|} \\ &= 100 \cdot 8.295 + 0.7773 \cdot 500 \cdot 7.689 \\ &= 3,817.95 \end{aligned}$$

Yet another way to write the present value of this annuity involves a decreasing arithmetic annuity:

$$500 \cdot a_{\overline{15}|} - 100 \cdot (Da)_{\overline{4}|}$$

The payment pattern is equivalent to a level 500 in all 15 years, reduced in the first 4 years by 400, 300, 200, and 100.

This expression can be evaluated as follows:

$$\begin{aligned} 500 \cdot \frac{1 - v^4}{i} - 100 \cdot \frac{4 - a_{\overline{4}|}}{i} &= 4,701.3344 - 883.3867 \\ &= 3,817.95 \end{aligned}$$

Example (2.100)

An annuity-immediate has 5 annual payments of 100, followed by a perpetuity of 200 starting in the 6th year. Find its present value at 8%.

Solution.
There are a number of ways to attack this problem. Perhaps the simplest is to think of this annuity as a perpetuity-immediate of 100 starting now, plus a second perpetuity-immediate of 100 starting in 5 years:

$$100 \cdot a_{\overline{\infty}|} + 100 \cdot v^5 \cdot a_{\overline{\infty}|}$$

Present Value	Payments							
100/0.08	100	100	100	100	100	100	100	...
$v^5 \cdot (100/0.08)$	0	0	0	0	0	100	100	...
time t 0	1	2	3	4	5	6	7	...

The present value of a single perpetuity of 100 is:

$$\frac{100}{0.08} = 1,250$$

Thus the total present value is:

$$1,250 + v^5 \cdot 1,250 = 2,100.73$$

Alternatively, we could treat this payment pattern as a perpetuity-immediate of 200, minus a 5-year annuity-immediate of 100:

$$\begin{aligned} 200 \cdot a_{\overline{\infty}|} - 100 \cdot a_{\overline{5}|} &= \frac{200}{0.08} - 100 \cdot \frac{1 - 1.08^{-5}}{0.08} \\ &= 2,100.73 \end{aligned}$$

Another approach is to treat the payments as a 5-year annuity of 100, plus a 5-year-deferred perpetuity of 200:

$$\begin{aligned} 100 \cdot a_{\overline{5}|} + 200 \cdot {}_{5|}a_{\overline{\infty}|} &= 100 \cdot \frac{1 - 1.08^{-5}}{0.08} + \frac{1}{1.08^5} \cdot \frac{200}{0.08} \\ &= 2,100.73 \end{aligned}$$

Exercise (2.101)

An annuity-immediate has a first payment of 100, and its payments increase by 100 each year until they reach 500. The remaining payments are a perpetuity-immediate of 500 beginning in year 6. Find the present value at 6.5%.

Answer: 6,808.92

Example (2.102)

Deposits are made at the beginning of each year for 25 years to an account that earns a 5% annual effective interest rate. The initial deposit (at time 0) is 1,000. Starting with the 2nd deposit and continuing through the 10th deposit, the amount of each deposit is 500 larger than the previous one. Then in the 11th through 15th years, each deposit is 200 *smaller* than the previous deposit. After 15 years, each of the remaining deposits is 6% *smaller* than the preceding deposit.

What will the account balance be at the end of 25 years?

Solution.
This is a very complex pattern of payments. There are various ways to analyze these payments, but any approach will involve at least 3 separate calculations (the first 10 years, years 11 through 15, and years 16 through 25). The value of the payments for each period must be determined as of the valuation date (time 25).

Here is one possible approach:

a) Use the PQ formula for the first 15 years, with $P = 1{,}000$ and $Q = 500$.

b) Subtract an increasing annuity for years 11-15, with an initial payment of 700. (This 700 annual decrease offsets the increases of 500 from a), creating a net annual decrease of 200 in this period.)

c) Add a deferred geometric (decreasing) annuity for years 16-25, with $g = -6\%$.

$$FV_{a)} = \left(1,000 \cdot \ddot{s}_{\overline{15|}} + 500 \cdot \frac{s_{\overline{15|}} - 15}{d}\right) \cdot 1.05^{10}$$

$$= \left(1,000 \cdot 22.6575 + 500 \cdot \frac{21.5786 - 15}{0.05/1.05}\right) \cdot 1.6289$$

$$= 149,422.43$$

$$FV_{b)} = -700 \cdot (I\ddot{s})_{\overline{5|}} \cdot 1.05^{10} = -700 \cdot \frac{\ddot{s}_{\overline{5|}} - 5}{d} \cdot 1.05^{10}$$

$$= -700 \cdot \frac{\frac{1.05^5 - 1}{0.05/1.05} - 5}{0.05/1.05} \cdot 1.05^{10}$$

$$= -19,201.60$$

$$FV_{c)} = (1,000 + 9 \times 500 - 5 \times 200) \cdot (1 - 0.06) \cdot \ddot{s}^{-6\%}_{\overline{10|}5\%}$$

$$= 4,230 \cdot \frac{1.05^{10} - 0.94^{10}}{(0.05 - (-0.06))/1.05}$$

$$= 44,022.51$$

$$\text{Total } = 149,422.43 - 19,201.60 + 44,022.51 = 174,243.34$$

Section 2.15 Annuities with Payments More Frequent than Annual

Thus far we have considered annuities that have annual payments or continuous payments. However, many annuities, such as loan payments or pensions, make payments at other intervals (e.g., monthly or quarterly). In this section, we will develop methods for annuities whose payments are more frequent than annual.

Interest rates for loans, including mortgage loans, are typically quoted as nominal rates. If a lender offers a monthly-payment mortgage at a 6% mortgage rate, it is probably a nominal annual rate of 6% convertible monthly, which is a monthly effective rate of 6% ÷ 12 = 0.5%. Knowing the monthly effective rate makes it easy to calculate the monthly payment amount.

Example (2.103)

Find the level monthly payment for a 30 year mortgage loan of 300,000 at an interest rate of 6% convertible monthly.

Solution.
The monthly effective interest rate is 0.5%, so we can simply use the usual annuity functions, recognizing that the period is 1 month instead of 1 year:

$$P = \frac{300,000}{a_{\overline{360}|0.5\%}} = \frac{300,000}{166.7916} = 1,798.65$$

The calculator solution is direct. Note that mortgage payments are made at the end of the month, so your calculator should be in END mode. The loan is for 360 months at a 0.5% monthly effective rate.

Set N = 360, I/Y = 0.5, and PV = 300,000. CPT PMT = −1,798.65.

Exercise (2.104)

Find the monthly payment for the loan in Example (**2.103**) if the term is 15 years.

Answer: 2,531.57

Example (2.105)

An annuity-immediate has 20 initial quarterly payments of 25 each, followed by a perpetuity of quarterly payments of 50 starting in the sixth year. Find the present value at 8% convertible quarterly.

Solution.
We can think of this annuity as a quarterly perpetuity-immediate of 25 starting now, augmented by a second quarterly perpetuity-immediate of 25 starting in 5 years.

		Payments							
$\text{PV} = 25/0.02$		25	25	25	⋯	25	25	25	⋯
$\text{PV} = v^{20} \cdot (25/0.02)$		0	0	0	⋯	0	0	25	⋯
Quarters	$t = 0$	1	2	3	⋯	19	20	21	⋯

The quarterly effective interest rate is 2%.

The present value of a single perpetuity of 25 is $\frac{25}{0.02} = 1,250$.

Thus the total present value is $1,250 + \frac{1,250}{1.02^{20}} = 2,091.21$.

There is actuarial notation for the present value of an annuity that makes m payments per year. The symbol $a_{\overline{n}|}^{(m)}$ represents the present value of an n-year annuity-immediate that pays $1/m$ at the end of each $1/m$ of a year. For example, consider an annuity that makes 12 payments per year. The symbol $a_{\overline{n}|}^{(12)}$ is the present value of an annuity that pays $1/12$ at the end of each month for n years. Note that the total amount paid each year is 1. The formula for the present value of an annuity that makes m payments per year can be developed as follows:

$$
\begin{aligned}
a_{\overline{n}|}^{(m)} &= \frac{1}{m} \cdot a_{\overline{m \cdot n}|\frac{i^{(m)}}{m}} \\
&= \frac{1}{m} \cdot \frac{1 - \left(1 + \frac{i^{(m)}}{m}\right)^{-m \cdot n}}{\frac{i^{(m)}}{m}} \\
&= \frac{1 - \left[\left(1 + \frac{i^{(m)}}{m}\right)^{m}\right]^{-n}}{i^{(m)}} \\
&= \frac{1 - v^n}{i^{(m)}} = \frac{i}{i^{(m)}} \cdot a_{\overline{n}|}
\end{aligned}
$$

This gives us the following formula for the present value of an n-year unit annuity-immediate that makes "m-thly" payments totaling 1 per year:

$$a_{\overline{n}|}^{(m)} = \frac{1 - v^n}{i^{(m)}} = \frac{i}{i^{(m)}} \cdot a_{\overline{n}|} \tag{2.106}$$

Note: The expression $\frac{1-v^n}{i^{(m)}}$ *in Formula* (**2.106**) *may* <u>*appear*</u> *to violate the rule that we never do calculations using nominal interest rates. (Clearly,* $i^{(m)}$ *is a nominal interest rate.) However, note that in the derivation of this formula (i.e., the equations just above Formula* (**2.106**)*), the expression after the second equals sign has a fraction whose denominator is* $\frac{i^{(m)}}{m}$*. This is the* m*-thly* <u>*effective*</u> *interest rate. This is, of course, proper. That fraction with* $\frac{i^{(m)}}{m}$ *in the denominator is then multiplied by* $\frac{1}{m}$*, which cancels with the* m *in* $\frac{i^{(m)}}{m}$*, leaving just* $i^{(m)}$ *in the denominator. So we are not actually performing a calculation with a nominal interest rate, but rather with an* m*-thly* <u>*effective*</u> *rate that has been multiplied by* m*. By contrast, an example of an* <u>*improper*</u> *use of a nominal rate is the accumulation factor* $\left(1 + i^{(4)}\right)^n$*, which is* <u>*never*</u> *correct.*

By analysis similar to the above, we can develop the following additional formulas for annuities with m-thly payments:

(2.107)
$$\ddot{a}^{(m)}_{\overline{n}|} = \frac{1-v^n}{d^{(m)}} = \frac{i}{d^{(m)}} \cdot a_{\overline{n}|} = \frac{d}{d^{(m)}} \cdot \ddot{a}_{\overline{n}|}$$

(2.108)
$$s^{(m)}_{\overline{n}|} = \frac{(1+i)^n - 1}{i^{(m)}} = \frac{i}{i^{(m)}} \cdot s_{\overline{n}|}$$

(2.109)
$$\ddot{s}^{(m)}_{\overline{n}|} = \frac{(1+i)^n - 1}{d^{(m)}} = \frac{i}{d^{(m)}} \cdot s_{\overline{n}|} = \frac{d}{d^{(m)}} \cdot \ddot{s}_{\overline{n}|}$$

These appear to be a lot of formulas to memorize, but if you see the pattern, you will find them easier to absorb. In fact, you can think of the standard annual-payment formulas $\left(\frac{1-v^n}{i}, \frac{1-v^n}{d}, \text{etc.}\right)$ as being special cases of the above formulas with $m = 1$. (By definition, $i^{(1)} = i$ and $d^{(1)} = d$, so the i or d in the denominator of an annual-payment annuity is actually $i^{(1)}$ or $d^{(1)}$.) At the opposite extreme, when payments are made continuously, m is infinite and the denominator is $i^{(\infty)} = d^{(\infty)} = \delta$. So the above formulas for m-thly annuities cover all payment frequencies, from annual ($m = 1$) to continuous ($m = \infty$).

If an n-year annuity has m equally spaced payments per year, and if the m payments within each year are equal in amount, then the annuity's present value (or future value) can be found by first calculating the value of an n-year *annual* payment annuity-immediate, and then multiplying that value by a factor of $\frac{i}{i^{(m)}}$ or $\frac{i}{d^{(m)}}$, depending on whether it is an m-thly annuity-immediate or an m-thly annuity due. The requirements are:

1. the m payments *within each year* must be *level*, and
2. the *total* of the m payments during each year must equal the *annual* payment in that year under the annual payment annuity-immediate whose value has been calculated.

In the case of continuous payments ($m = \infty$), calculate the value of the corresponding annuity-immediate and multiply by $\frac{i}{\delta}$. In this case, the continuous *rate* of payment within each year must be level, and the total of the continuous payments within each year must equal the annual payment of the annuity-immediate whose value was calculated.

This adjustment methodology is particularly useful when working with arithmetic or geometric increasing or decreasing annuities where the increases or decreases occur annually but the payments *within* each year are *level*. In the case of geometric annuities, the denominator of the annual-payment geometric annuity-immediate formula is not simply i. Nonetheless, we can still apply an adjustment factor of $\frac{i}{i^{(m)}}$ or $\frac{i}{d^{(m)}}$ or $\frac{i}{\delta}$ to the value calculated using annual payments.

This adjustment technique can also be used with level annuities, but it is generally easier to calculate the present value of a level annuity by finding the *effective interest rate per payment period* and then applying the standard annuity formulas (with one payment per interest conversion period). For example, if a 10-year level-payment annuity has quarterly payments, find the quarterly effective rate and treat the annuity as a 40-period annuity at that interest rate. This method was used in Examples (**2.103**) and (**2.105**) at the beginning of this section.

The following example applies the formula for an $m$thly annuity, but also shows how to solve the problem using the adjustment technique and how to solve it using the effective interest rate per period.

Example (2.110)

A saver deposits 100 into a bank account at the end of every month for 10 years. If the account earns interest at an annual effective rate of 6%, what is the saver's balance at the end of 10 years?

Solution.
Using Formula (**2.108**), we can calculate the accumulated value of a 10-year annuity-immediate with monthly payments:

$$s_{\overline{10}|6\%}^{(12)} = \frac{(1+i)^{10}-1}{i^{(12)}} = \frac{1.06^{10}-1}{12\cdot\left(1.06^{1/12}-1\right)} = 13.539$$

This is the value at time 10 of monthly deposits that total 1 each year for 10 years. But in this problem, the saver is depositing a total of 1,200 each year, so we multiply this value by 1,200:

$$Bal_{10} = 1,200\cdot(13.539) = 16,247.34$$

We can also solve this problem by calculating the accumulated value based on *annual* payments, and then adjusting that value to reflect *monthly* payments:

$$1,200\cdot s_{\overline{10}|6\%} = 1,200\cdot\frac{1.06^{10}-1}{0.06} = 15,816.95$$

$$i/i^{(12)} = 0.06/\left[12\cdot\left(1.06^{1/12}-1\right)\right] = 1.02721$$

$$Bal_{10} = 1,200\cdot s_{\overline{10}|6\%}^{(12)} = 1,200\cdot s_{\overline{10}|6\%}\cdot\frac{i}{i^{(12)}}$$

$$= 15,816.95\times 1.02721 = 16,247.34$$

Alternatively, we can calculate the *monthly* effective rate and find the value of a 120-period annuity-immediate with payments of 100 per period:

$$\text{monthly effective rate } = 1.06^{1/12}-1 = 0.004868 = 0.4868\%$$

$$Bal_{10} = 100\cdot s_{\overline{120}|0.4868\%} = 100\cdot\frac{1.004868^{120}-1}{0.004868} = 16,247.34$$

Exercise (2.111)

A car loan of 20,000 is to be repaid by monthly payments over a 4-year period. Payments are made at the end of each month, with the first payment being due one month after the loan amount was received. If the loan is based on a nominal rate of 6.6% convertible monthly, what is the amount of each monthly payment?

Answer: 475.22

Section 2.16 Annuities with Payments Less Frequent than Annual

In some cases, an annuity or perpetuity may have payments that occur less frequently than annually. For example, payments could be made every second year or every fifth year. As was the case for annuities with more frequent payments, there are formulas for calculating the present value or future value of these annuities with less frequent payments. These formulas involve finding the ratio of two annuity values, as demonstrated in the following example.

Example (2.112)

An annuity pays 1 at the end of each 5 years for 40 years (a total of 8 payments). The present value of this annuity-immediate at an annual effective rate of 6% can be calculated as follows:

$$\frac{a_{\overline{40}|6\%}}{s_{\overline{5}|6\%}} = \frac{\left(1 - 1.06^{-40}\right)/0.06}{\left(1.06^{5} - 1\right)/0.06} = \frac{15.0463}{5.6371}$$

$$= 2.6692$$

The formula for this particular annuity has $a_{\overline{40}|6\%}$ in the numerator because we are calculating the present value of an annuity with a term of 40 years. It has $s_{\overline{5}|6\%}$ in the denominator because the payments occur at the end of each 5-year period.

To understand why this formula produces the correct value for a 40 year annuity paying 1 at the end of every 5^{th} year, we will analyze its components.

$a_{\overline{40}|6\%} = 15.0463$ is the value of a payment of 1 at the end of each year for 40 years. We can also think of these payments as a series of eight 5-year annuities, each of which pays 1 at the end of every year for 5 years. And each of those 5-year annuities could be replaced by a single payment of amount $s_{\overline{5}|6\%} = 5.6371$ made at the end of the 5 years without changing the present value of the 40-year annuity.

Thus 40 annual payments of 1 have the same present value as 8 payments of 5.6371 that are made at the end of each 5 years for 40 years. Since we want to find the value of 8 payments of 1 (not 5.6371), we need to divide $a_{\overline{40}|6\%}$ by 5.6371. That is, we need to divide the present value of 40 payments of 1 ($= a_{\overline{40}|6\%}$) by the accumulated value of 5 payments of 1 ($= s_{\overline{5}|6\%}$). The result is the present value of a 40-year annuity that pays 1 at the end of every 5^{th} year.

Exercise (2.113)

A 20-year annuity immediate makes a payment of 100 at the end of each 4 years (a total of 5 payments). At an annual effective interest rate of 5%, what is the present value of this annuity?

Answer: 289.14

The formula used in the preceding example and exercise for the present value of annuities-immediate can be generalized to handle future values and annuities-due:

(2.114)

$$\text{Present value of an annuity-immediate} = \frac{a_{\overline{n}|i}}{s_{\overline{m}|i}}$$

$$\text{Present value of an annuity-due} = \frac{a_{\overline{n}|i}}{a_{\overline{m}|i}}$$

$$\text{Future value of an annuity-immediate} = \frac{s_{\overline{n}|i}}{s_{\overline{m}|i}}$$

$$\text{Future value of an annuity-due} = \frac{s_{\overline{n}|i}}{a_{\overline{m}|i}}$$

where:

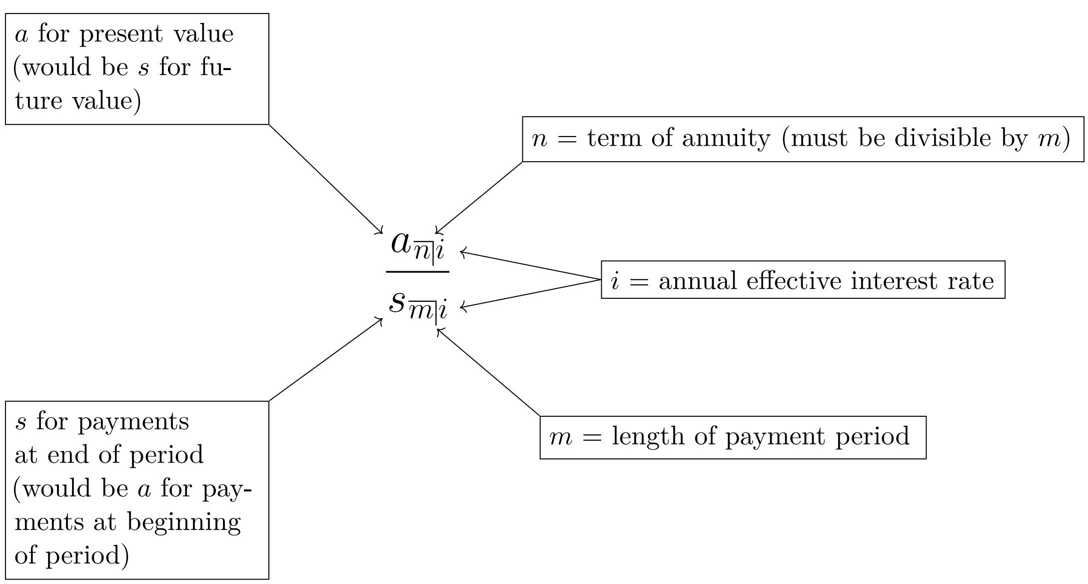
$n =$ term of the annuity in years

$m =$ length of the payment period in years

$i =$ annual effective interest rate

Note: n must be divisible by m.

The rules for forming these ratios are summarized in the following diagram. It contains the formula for the *present* value of an n-year annuity with a payment of 1 at the *end* of every m years, and indicates how that formula would be modified to calculate a *future* value or the value of an annuity-*due*.



Example (2.115)

An annuity will make a payment of 1,000 today, and a payment of 1,000 every 4 years until a total of 10 payments have been made. Based on an annual effective rate of 5%, we will calculate the value of this annuity today, and also as of the date of the 10^{th} payment.

This annuity will make payments at the beginning of each 4-year period for 40 years, so it is an annuity-due. To calculate its present value, we use the formula for the present value of an annuity-due whose payments are less frequent than annual:

$$1,000 \cdot \frac{a_{\overline{40}|5\%}}{a_{\overline{4}|5\%}} = 1,000 \cdot \frac{17.15908}{3.54595} = 4,839.07$$

To find the annuity's value as of its last payment, we need to recognize that this is *not* the future value of an annuity-due. An annuity-due makes payments at the *beginning* of each period, and its future value is the value calculated as of the *end* of the last payment period, which is *not a payment date*. Since we are calculating a value as of the date of the last payment, we need to think of this as an annuity-immediate (which began 4 years before the first payment, so that its first payment is also an end-of-period payment).

Using the formula for the future value of an annuity-immediate:

$$1,000 \cdot \frac{s_{\overline{40}|5\%}}{s_{\overline{4}|5\%}} = 1,000 \cdot \frac{120.7998}{4.3101} = 28,026.98$$

Exercise (2.116)

An annuity pays 100 every second year for 10 years. Based on a 4.5% annual effective interest rate, calculate the future value of this annuity (as of time 10) if it is:

(a) an annuity-immediate

(b) an annuity-due

Answers: (a) 600.89 (b) 656.19

Example (2.117)

Minnie Cooper borrows 20,000 at an annual effective rate of 8%, and agrees to repay the loan over 20 years with level payments at the end of every second year. What is the amount of Minnie's biennial payment?

Solution:
The equation of value for this loan is:

$$20,000 = P \cdot \frac{a_{\overline{20}|8\%}}{s_{\overline{2}|8\%}} \longrightarrow P = 20,000 \div \frac{\left(1 - 1.08^{-20}\right)/0.08}{\left(1.08^2 - 1\right)/0.08} = 4,237.05$$

Exercise (2.118)

Joe King is saving for retirement. Because he receives a large bonus every 3 years, he plans to contribute to his retirement savings at the end of every third year for the next 30 years. He estimates that his retirement savings account will accumulate at a 6% annual effective rate.

If Joe's retirement savings goal is 1,000,000, how much should he deposit every 3 years in order to reach this goal?

Answer: 40,269.07

You may have noticed that all of the formulas for annuities payable less frequently than annually (the formulas in **(2.114)***) use only annuity-immediate functions. There are no annuity-due functions in these formulas. We could write these same formulas using annuities-due, and the calculated values would be the same, but both numerator and denominator must use the same type of annuity function (either immediate or due).*

About the Practice Exams

After learning the material in each module, and after reviewing the modules and midterm exams in this manual, you will be ready to tackle these practice exams. Like the Society of Actuaries' Exam FM, each of these practice exams consists of 30 questions on the topics in the FM syllabus. As in SOA Exam FM, if your answer for a numerical question does not match any of the answer choices exactly, you should select the choice that is closest to your calculated answer.

The 14 practice exams fall into 3 categories:

a) The first 6 practice exams are relatively straightforward, to enable you to review the basics of each topic. You may want to attempt them in a *non-timed* environment to evaluate your skills and understanding.

b) The next 5 practice exams introduce more difficult questions in order to replicate the actual exam experience. You should take each of these in a *timed* environment, allowing yourself 2.5 hours to complete each exam. This will give you experience with exam-like conditions.

c) The final 3 practice exams include especially challenging problems. Try to finish these exams within 2.5 hours, but you may need extra time due to the difficulty of the problems. The important thing is to be sure you understand the solutions, so that you will be able to apply these methods when you take Exam FM.

Please keep in mind that the actual exam questions are confidential, and there is no guarantee that the questions you encounter on Exam FM will look exactly like those included in these practice exams.

Practice Exam 1

Questions

1. You are given the following yield curve:

Year	Spot Rate
1	5.5%
2	5.2%
3	5.0%
4	4.4%
5	4.0%

What is the 4-year forward rate ($i_{4,5}$)?

(A) 2.2% (B) 2.3% (C) 2.4% (D) 2.5% (E) 2.6%

2. Find the Macaulay duration of a 10-year, 1,000 par value bond with 8% annual coupons, based on an annual effective yield of 6.5%.

(A) 7.2 (B) 7.4 (C) 7.6 (D) 7.8 (E) 8.0

3. At an annual effective loan interest rate i, a loan of K can be repaid in either of two ways:

i) 475 now and 475 in 1 year, or

ii) 570 in 2 years and 570 in 3 years.

Calculate K.

(A) 893 (B) 901 (C) 909 (D) 917 (E) 925

4. A 10-year annuity-immediate pays 100 quarterly for the first year. In each subsequent year, the quarterly payment amount is increased by 5% over the payment amount during the previous year. At a nominal annual interest rate of 8% convertible quarterly, what is the present value of this annuity?

(A) 2,997 (B) 3,075 (C) 3,108 (D) 3,225 (E) 3,333

5. At an annual effective interest rate i, the present value of a 10-year annuity-immediate with level annual payments is X. At the same interest rate, a 20-year annuity-immediate with the same annual payment amount has a present value of $1.5X$. Calculate i.

(A) 7.2% (B) 7.4% (C) 7.6% (D) 7.8% (E) 8.0%

6. A 10-year bond that pays 4.3% annual coupons has a price equal to its face amount. (In other words, it is a "par bond.") A similar 10-year bond has the same yield rate and has a price (at issue) of 104 per 100 of face amount. What is this second bond's annual coupon rate?

☐ (A) 4.4% ☐ (B) 4.5% ☐ (C) 4.6% ☐ (D) 4.7% ☐ (E) 4.8%

7. Spot rates for terms of 1 to 4 years are as follows:

Term (in years)	1	2	3	4
Spot Rate	5.0%	5.75%	6.25%	X

The coupon rate for a 4-year par bond is 6.62%. Calculate X.

☐ (A) 6.60% ☐ (B) 6.65% ☐ (C) 6.70% ☐ (D) 6.75% ☐ (E) 6.80%

8. A company has liabilities requiring payments of 1,000; 3,000; and 5,000 at the end of years 1, 2 and 3, respectively. The investments available to the company are the following zero-coupon bonds:

Maturity (years)	Annual Effective Yield	Par Value
1	7%	1,000
2	8%	1,000
3	9%	1,000

Determine the cost to match the company's liability cash flows exactly.

☐ (A) 6,918 ☐ (B) 7,024 ☐ (C) 7,165 ☐ (D) 7,368 ☐ (E) 7,522

9. Pete Moss creates a retirement fund by making deposits at the end of each month for 20 years. For the first 10 years he deposits 100 per month, and for the last 10 years he deposits 200 per month. The fund earns interest at a nominal annual rate of 6% convertible monthly. At the end of 20 years, Pete uses the proceeds to purchase a 30-year annuity-immediate with monthly payments. The annuity is priced based on a nominal rate of 8% convertible monthly. What is the amount of Pete's monthly payment from this annuity?

☐ (A) 408 ☐ (B) 425 ☐ (C) 437 ☐ (D) 441 ☐ (E) 459

10. An annuity makes annual payments at the beginning of each year for 20 years. For the first 10 years the payments are 100. Starting with the 11^{th} payment, each payment is increased by 6% over the previous payment. At an annual effective rate of 8%, what is the present value of this annuity?

☐ (A) 1,177 ☐ (B) 1,190 ☐ (C) 1,202 ☐ (D) 1,213 ☐ (E) 1,225

11. An annual-coupon corporate bond has an annual effective yield of 7.2% at its current price of 972.48. At 7.2%, the bond's Macaulay duration is 7.1245. Using the first-order modified approximation method, estimate the change in price that would cause the bond's yield to increase by 0.10%.

(A) −6.463 (B) −6.685 (C) −6.814 (D) −7.012 (E) −7.163

12. A 40-year loan is repaid by level annual payments at the end of each year. The principal paid in the 20^{th} payment is 162.43 and the principal paid in the 25^{th} payment is 238.66. Find the amount that was borrowed.

(A) 9,500 (B) 9,750 (C) 10,000 (D) 10,250 (E) 10,500

13. Thelma deposits 100 into an account at the end of each year for 20 years. Her account earns interest at an annual effective rate of 5%. Louise deposits money into an account at the end of each year for 20 years. Her account also earns interest at an annual effective rate of 5%. The amount of Louise's deposit increases each year in the following pattern: $P, 2P, 3P, \ldots, 20P$. At the end of 20 years the balances in the two accounts are equal. Calculate P.

(A) 10.93 (B) 11.05 (C) 11.12 (D) 11.23 (E) 11.35

14. Dinah Soares borrows money to buy a new piano. She agrees to pay back the loan with level annual payments at the end of each year for 30 years. The annual effective interest rate is 7%. The amount of interest in her 10^{th} payment is 366.74. What is the amount of interest in her 20^{th} payment?

(A) 221.86 (B) 229.64 (C) 244.18 (D) 250.72 (E) 253.80

15. Rita Booke makes a deposit into an account. For the first 5 years the account accumulates at a force of interest of 0.05. For the next 10 years the fund accumulates at a nominal annual rate of discount of 6% convertible quarterly. For the 15-year period, what is the equivalent nominal annual interest rate convertible monthly?

(A) 5.59% (B) 5.71% (C) 5.83% (D) 5.96% (E) 6.04%

16. Robin Banks purchases a 10-year 1,000 par value bond that pays semi-annual coupons at an 8% annual rate. The bond is priced to yield 7.5% convertible semi-annually. Robin reinvests the coupon payments in a fund that pays a nominal rate of 7% convertible semi-annually. Over the 10-year period, what is Robin's nominal annual yield convertible semi-annually?

(A) 7.36% (B) 7.41% (C) 7.48% (D) 7.56% (E) 7.63%

17. You are given the following yield curve:

Year	Spot Rate
1	4.0%
2	4.2%
3	4.6%
4	—
5	5.1%

Given that the 4-year forward rate ($i_{4,5}$) is 6.1%, calculate the 4-year spot rate (s_4).

(A) 4.81% (B) 4.83% (C) 4.85% (D) 4.87% (E) 4.89%

18. A 20-year annuity-immediate has annual payments. The first payment is 100 and subsequent payments increase by 100 each year until they reach 1,000. Each of the remaining payments is 1,000. At an annual effective interest rate of 7.5%, what is the present value of this annuity?

(A) 6,201 (B) 6,372 (C) 6,413 (D) 6,584 (E) 6,700

19. Vera Phyde buys a 1,000 par value 5-year zero-coupon bond priced to yield a 6% annual effective interest rate. At the same time she buys a 1,000 par value 5-year bond with 8% semiannual coupons that is priced to yield 7% convertible semi-annually. The coupon payments are reinvested at 6.5% convertible semi-annually. Over their 5-year term, what is Vera's annual effective yield from the combination of these investments?

(A) 6.0% (B) 6.2% (C) 6.4% (D) 6.6% (E) 6.8%

20. Account A earns compound interest at an annual effective rate i (where i is greater than 0). Account B earns compound interest at an annual effective rate equal to $1.1 \cdot i$.

1,000 is deposited into each of these accounts at $t = 0$. No other deposits or withdrawals occur. At the end of 20 years (at $t = 20$), the balance in Account B will be 10% larger than the balance in the Account A.

What will be the difference between the balances in the two accounts at the end of 10 years (at $t = 10$)?

(A) 75 (B) 80 (C) 85 (D) 90 (E) 95

21. A 20-year monthly-payment variable-rate mortgage has an initial principal of 200,000 and an initial interest rate of 3.6% convertible monthly.

At the end of 2 years, the mortgage interest rate increases from 3.6% to 4.5% and a new monthly payment amount (for the 3^{rd} and later years) is calculated, based on the new interest rate and the outstanding loan balance on that date. There are no other interest rate changes during the first 5 years of the loan.

To the nearest 100, what is the outstanding balance of the loan at the end of 5 years (immediately after the 60^{th} monthly payment)?

(A) 162,600 (B) 162,900 (C) 164,000 (D) 164,300 (E) 165,400

22. A 20-year bond has a face value (and maturity value) of 1,000. It pays semi-annual coupons at a 6% (annual) coupon rate. The bond is callable on any coupon date on or after its 10^{th} anniversary, with a 5% call premium.

If an investor purchases this bond on its issue date at a price of 1,060, and holds the bond until it matures or is called, what is the minimum yield the investor could earn (expressed as a nominal rate, convertible semi-annually)?

(A) 5.50% (B) 5.59% (C) 5.76% (D) 5.96% (E) 6.37%

23. Amanda makes a deposit of 1,000 at the end of each month for 30 years into an account that earns an annual effective interest rate i. Beginning one month after the last deposit, she can make perpetual monthly withdrawals of 4,000 from the same account at the same interest rate.

Calculate i.

(A) 5.3% (B) 5.5% (C) 5.7% (D) 5.8% (E) 6.0%

24. Charlotte takes out a 30-year mortgage for an amount A at an interest rate i convertible monthly. She notices that if she makes all the monthly payments for 30 years, the total amount of her payments will equal 3 times the amount she borrowed. What percentage of the loan principal will Charlotte repay during the first 2 years of the loan?

(A) 1.32% (B) 1.38% (C) 1.42% (D) 1.46% (E) 1.49%

25. Barb Dwyer invests 500 at $t = 0$ at a nominal annual interest rate of 6% convertible quarterly. What additional amount will Barb need to invest at $t = 2$ in order to have a total of 1,000 at $t = 5$?

(A) 242 (B) 273 (C) 278 (D) 290 (E) 327

26. Lisa Carr purchased a newly-issued 25-year bond that pays semi-annual coupons at an 8% (annual) rate. The bond has a par value (and a redemption value) of 1,000. The bond is callable on or after its 10th anniversary with a 10% call premium (i.e., 1,100 is payable if it is called).

Lisa purchased this bond at issue at a price that will assure her a rate of return of at least 7.5% (a nominal rate, convertible semi-annually). If the bond is called on its 12th anniversary, what is Lisa's actual rate of return?

☐ (A) 7.5% ☐ (B) 7.6% ☐ (C) 7.7% ☐ (D) 7.8% ☐ (E) 7.9%

27. A 20-year bond has a par value of 1,000 and a maturity value of 1,040. It pays semi-annual coupons at a 5% annual rate. Calculate the bond's purchase price if its yield to maturity is 6% convertible monthly.

☐ (A) 876 ☐ (B) 880 ☐ (C) 884 ☐ (D) 889 ☐ (E) 897

28. The following table gives the term structure of spot interest rates.

Term (in years)	1	2	3	4
Spot interest rate	4.25%	4.80%	X	5.40%

If the two-year forward rate ($i_{2,3}$) is 6.5%, what is the value of X?

☐ (A) 4.95% ☐ (B) 5.04% ☐ (C) 5.20% ☐ (D) 5.36% ☐ (E) 11.61%

29. Sarah Naide takes out a loan for 50,000 with 30 quarterly payments. For the first 10 payments, Sarah will pay only the interest due at the end of each quarter. For the last 20 payments, Sarah will pay X at the end of each quarter (an amount that will pay off the loan). If the annual effective interest rate for the loan is 5.8%, what is the total of all of Sarah's payments for this loan?

☐ (A) 50,709 ☐ (B) 57,785 ☐ (C) 64,882 ☐ (D) 65,330 ☐ (E) 67,050

30. On January 1, Patty O. opens a savings account. She deposits 300 at the beginning of odd months (January, March, etc.) and withdraws 250 at the beginning of even months (February, April, etc.). If Patty earns an interest rate of 6.3% convertible monthly, what is the value of her fund at the end of 12 months? (Assume 30-day months.)

☐ (A) 310 ☐ (B) 319 ☐ (C) 328 ☐ (D) 337 ☐ (E) 349

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Solutions

Reset the Exam

Question #	Answer
1	C
2	B
3	C
4	E
5	A
6	E
7	C
8	D
9	E
10	A
11	A
12	B
13	D
14	E
15	B

Question #	Answer
16	A
17	C
18	E
19	D
20	B
21	D
22	A
23	B
24	A
25	B
26	D
27	D
28	D
29	C
30	B

1. The four-year forward rate ($i_{4,5}$) is given by:

$$1+i_{4,5} = (1+s_5)^5/(1+s_4)^4 = 1.04^5/1.044^4 = 1.02415$$
$$i_{4,5} = 0.024$$

(*Note the unusually low value, due to the inverted yield curve.*)

Answer C

2. $D_{\text{mac}} = \dfrac{80(Ia)_{\overline{10|}} + 10 \times 1,000 \cdot v^{10}}{\text{Bond Price}}$

$v^{10} = 1.065^{-10} = 0.532726 \qquad \ddot{a}_{\overline{10|}} = 7.6561 \qquad a_{\overline{10|}} = 7.1888$

$(Ia)_{\overline{10|}} = \frac{\ddot{a}_{\overline{10|}} - 10 \cdot v^{10}}{i} = 35.8284$

Bond Price $= 80 \cdot a_{\overline{10|}} + 1,000 \cdot v^{10} = 1,107.83$

Or set N = 10, PMT = 80, I/Y =6.5, FV = 1,000 and CPT PV = −1,107.83.

$D_{\text{mac}} = (80 \times 35.8284 + 10,000 \times 0.532726)/1,107.83 = 7.396$

Answer B

3. $K = 475 + 475 \cdot v = 570 \cdot v^2 + 570 \cdot v^3 = v^2 \cdot (570 + 570 \cdot v)$

$475 \cdot (1+v) = v^2 \cdot 570 \cdot (1+v) \rightarrow v^2 = 475/570 = 0.8333$

$v = 0.8333^{0.5} = 0.91287$

Note that each payment of 570 occurs 2 years after each payment of 475, so we could have written: $570v^2 = 475 \qquad v = (475/570)^{0.5} = 0.91287$

$K = 475(1+v) = 475(1.91287) = 908.61$

Answer C

4. The accumulated value of the first year's payments at the end of year 1 is 412.16.
(Set N = 4, I/Y = 2, PMT = 100, PV =0. CPT FV = −412.16.)

Thus the value of the annuity is the same as that of a 10-year annuity-immediate with annual payments, the first being 412.16 and subsequent payments increasing by 5% each year. The annual effective interest rate is:

$$i = (1.02)^4 - 1 = 0.08243$$

The present value of this annuity is:

$$412.16 \cdot a^{5\%}_{\overline{10|}8.2432\%} = 412.16 \cdot \frac{1-(1.05/1.08243)^{10}}{0.08243 - 0.05} = 3,333.28$$

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