TIME VALUE OF MONEY

Accumulation Factor $A(t_1, t_2)$: The accumulation at time t_2 of an investment of 1 at time t_1 for $t_1 < t_2$, $A(n) = A(0, n)$

Simple Interest: $A(n) = C(1 + in)$

Compound Interest: $A(n) = C(1 + i)^n$ $\quad i_n = i$

Present Value Factor: $v(n) = \dfrac{1}{A(n)}$

Effective Interest Rate: $i_n = \dfrac{A(n) - A(n-1)}{A(n-1)}$

Effective Discount Rate: $d_n = \dfrac{A(n) - A(n-1)}{A(n)}$

Simple Discount: $C(1 - nd)$

Effective Discount: $C(1 - d)^n$

Discount Rate: $d = \dfrac{i}{1+i} = 1 - v = iv$ $\quad v = (1+i)^{-1}$ $\quad \dfrac{1}{d} - \dfrac{1}{i} = 1$

Nominal Rates: $\left(1 + \dfrac{i^{(p)}}{p}\right)^p = 1 + i$ $\quad \left(1 - \dfrac{d^{(p)}}{p}\right)^p = 1 - d$

$i^{(p)} = p\left[(1+i)^{\frac{1}{p}} - 1\right]$ $\quad d^{(p)} = p\left[1 - (1-d)^{\frac{1}{p}}\right]$

Force of Interest: $\delta_t = \dfrac{V_t'}{V_t} = \dfrac{d}{dt}\ln V_t$ $\quad A(t_1, t_2) = e^{\int_{t_1}^{t_2} \delta_t dr}$ $\quad A(0, n) = e^{\int_0^n \delta dt} = e^{\delta n}$

Constant Force of Interest: $e^\delta = 1 + i$ $\quad \delta = \ln(1+i)$ $\quad v = e^{-\delta}$

PV of \$1 Due in t Years: $PV = (1+i)^{-t} = v^t = e^{-\delta t} = (1-d)^t = \left(1 + \dfrac{i^{(m)}}{m}\right)^{-mt} = \left(1 - \dfrac{d^{(m)}}{m}\right)^{mt}$

AV at time t of \$1 invested at time 0: $AV = (1+i)^t = e^{\delta t} = (1-d)^{-t} = \left(1 + \dfrac{i^{(m)}}{m}\right)^{mt} = \left(1 - \dfrac{d^{(m)}}{m}\right)^{-mt}$

Principle of Consistency: $A(t_0, t_n) = A(t_0, t_1)\, A(t_1, t_2) \ldots A(t_{n-1}, t_n)$

CASHFLOWS

Discrete cashflows: $c_{t_1} v(t_1) + c_{t_2} v(t_2) + \cdots + c_{t_n} v(t_n) = \sum_{j=1}^{n} c_{t_j} v(t_j)$

$\sum_{j=1}^{\infty} c_{t_j} v(t_j)$, if the No. of payments is infinite

$\rho(t)$: The rate of payment at time t per unit time.

$M(t)$: The total payment made between time 0 and time t

Continuously payable cashflows:

$\rho(t) = M'(t)$ for all t

The payment received between time α and time β:

$M(\beta) - M(\alpha) = \int_\alpha^\beta M'(t)dt = \int_\alpha^\beta \rho(t)dt$ where $0 \le \alpha < \beta \le T$

PV of the entire cashflow: $\int_0^T v(t)\rho(t)dt$ $\quad \int_0^\infty v(t)\rho(t)dt$, if the No. of payments is infinite

General Cashflow: $\sum c_t v(t) + \int_0^\infty v(t)\rho(t)dt$

Value at time t_1 of C due at time t_2: $C \exp\left[-\int_{t_1}^{t_2} \delta(t)dt\right]$ and $\int_{t_1}^{t_2} \delta(t)dt = \int_0^{t_2} \delta(t)dt - \int_0^{t_1} \delta(t)dt$

Valuing cashflows: $\sum c_t v(t) + \int_{-\infty}^{\infty} \rho(t)v(t)dt$

$$\begin{bmatrix}\text{Value at time } t_1 \\ \text{of cashflow}\end{bmatrix} = \begin{bmatrix}\text{Value at time } t_2 \\ \text{of cashflow}\end{bmatrix}\left[\frac{v(t_2)}{v(t_1)}\right]$$

$$\begin{bmatrix}\text{Value at time } t \\ \text{of cashflow}\end{bmatrix} = \begin{bmatrix}\text{Value at the present} \\ \text{time of cashflow}\end{bmatrix}\left[\frac{1}{v(t)}\right]$$

Interest income: $I(T) = \int_0^T C\delta(t)dt$

Capital C: $C = C\int_0^T \delta(t)v(t)dt + Cv(T)$

ANNUITY

Annuity-Immediate: $a_{\overline{n}|}$: one period before first payment, $s_{\overline{n}|}$: at time of last payment

$$a_{\overline{n}|} = v + v^2 + \cdots + v^n = \frac{1-v^n}{i}$$

$$s_{\overline{n}|} = 1 + (1+i) + \cdots + (1+i)^{n-1} = \frac{(1+i)^n - 1}{i} = a_{\overline{n}|}(1+i)^n$$

Annuity-Due: $\ddot{a}_{\overline{n}|}$: at time of first payment, $\ddot{s}_{\overline{n}|}$: one period after last payment

$$\ddot{a}_{\overline{n}|} = 1 + v + \cdots + v^{n-1} = \frac{1-v^n}{d} = (1+i)a_{\overline{n}|} = \left(\frac{i}{d}\right)a_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

$$\ddot{s}_{\overline{n}|} = (1+i) + (1+i)^2 + \cdots + (1+i)^n = \frac{(1+i)^n - 1}{d} = \ddot{a}_{\overline{n}|}(1+i)^n$$

$$= (1+i)s_{\overline{n}|} = \left(\frac{i}{d}\right)s_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

Continuous Annuity:

$$\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta} = \left(\frac{i}{\delta}\right)a_{\overline{n}|} = \int_0^n e^{-\delta t}dt$$

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta} = \left(\frac{i}{\delta}\right)s_{\overline{n}|} = \int_0^n e^{\delta(n-t)}dt = \bar{a}_{\overline{n}|}(1+i)^n$$

p-thly Annuity:

$$a_{\overline{n}|}^{(p)} = \frac{i}{i^{(p)}}a_{\overline{n}|} = \frac{1-v^n}{i^{(p)}} = v^{1/p}\ddot{a}_{\overline{n}|}^{(p)} \qquad \ddot{a}_{\overline{n}|}^{(p)} = \frac{i}{d^{(p)}}a_{\overline{n}|} = \frac{1-v^n}{d^{(p)}}$$

$$s_{\overline{n}|}^{(p)} = \frac{(1+i)^n - 1}{i^{(p)}} = \frac{i}{i^{(p)}}s_{\overline{n}|} = a_{\overline{n}|}^{(p)}(1+i)^n \qquad \ddot{s}_{\overline{n}|}^{(p)} = \frac{i}{d^{(p)}}s_{\overline{n}|} = \ddot{a}_{\overline{n}|}^{(p)}(1+i)^n$$

$$a = \sum_{t=1}^n X_t v^t \qquad a^{(p)} = \frac{i}{i^{(p)}}a$$

Non-integer values of n:

$$a_{\overline{n}|}^{(p)} = \frac{1}{p}\left(v^{1/p} + v^{2/p} + v^{3/p} + \cdots + v^{r/p}\right) = \frac{1}{p}\left[\frac{1-v^{r/p}}{(1+i)^{1/p} - 1}\right]$$

$$a_{\overline{n}|}^{(p)} \text{ at rate } i = \frac{1}{p}a_{\overline{np}|} \text{ at rate } i^{(p)}/p$$

$$a_{\overline{n}|}^{(p)} = a_{\overline{n_1}|}^{(p)} + fv^n \text{ where } n = n_1 + f$$

Perpetuity:

$$a_{\overline{\infty}|} = \frac{1}{i} \qquad \ddot{a}_{\overline{\infty}|} = \frac{1}{d} \qquad \bar{a}_{\overline{\infty}|} = \frac{1}{\delta}$$

$$a_{\overline{\infty}|}^{(p)} = \frac{1}{i^{(p)}} \qquad \ddot{a}_{\overline{\infty}|}^{(p)} = \frac{1}{d^{(p)}}$$

Deferred Annuity:

$$_{m|}a_{\overline{n}|} = v^{m+1} + v^{m+2} + v^{m+3} + \cdots + v^{m+n} = {}_{m+1|}\ddot{a}_{\overline{n}|} = v^m a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$$

$$_{m|}\ddot{a}_{\overline{n}|} = v^m \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|}$$

$$_{m|}\bar{a}_{\overline{n}|} = \int_m^{m+n} e^{-\delta t}dt = \bar{a}_{\overline{m+n}|} - \bar{a}_{\overline{m}|} = v^m \bar{a}_{\overline{n}|}$$

$$_{m|}a_{\overline{n}|}^{(p)} = v^m a_{\overline{n}|}^{(p)} \qquad _{m|}\ddot{a}_{\overline{n}|}^{(p)} = v^m \ddot{a}_{\overline{n}|}^{(p)} \qquad _{m|}(Ia)_{\overline{n}|} = v^m (Ia)_{\overline{n}|}$$

Increasing Annuity:

$$(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \cdots + nv^n = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}$$

$$(I\ddot{a})_{\overline{n}|} = 1 + 2v + 3v^2 + \cdots + nv^{n-1} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} = 1 + a_{\overline{n-1}|} + (Ia)_{\overline{n-1}|}$$

$$(I\bar{a})_{\overline{n}|} = \sum_{r=1}^{n}\left(\int_{r-1}^{r} rv^t dt\right) = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}$$

$$(\overline{Ia})_{\overline{n}|} = \int_0^n tv^t dt = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta}$$

TERM STRUCTURE OF INTEREST RATES

Factors causing interest rates to vary over time:

- Supply and demand
- Base rates
- Interest rates in other countries
- Expected future inflation
- Risk associated with changes in interest rates
- Tax rates

Spot Rates: n-year spot Rate of interest: y_n

Price of a n-year zero-coupon Bond: $P_n = (1+y_n)^{-n} \Rightarrow (1+y_n) = P_n^{-\frac{1}{n}}$

Term structure of interest rates: The variation by term of interest rates

Forward Rates: (Annual Effective)

Forward rate for the period $(t, t+1)$: $f_{t,1} = \frac{(1+r_{t+1})^{t+1}}{(1+r_t)^t} - 1 = \frac{P_t}{P_{t+1}} - 1$

Forward rate for the period $(t, t+r)$: $(1+y_t)^t(1+f_{t,r})^r = (1+y_{t+r})^{t+r} = P_{t+r}^{-1}$

$$(1+f_{t,r})^r = \frac{(1+y_{t+r})^{t+r}}{(1+y_t)^t} = \frac{P_t}{P_{t+r}}$$

Continuous time spot rates: t-year spot force of interest is Y_t

$$P_t = e^{-Y_t t} \Rightarrow Y_t = -\frac{1}{t}\log P_t$$

Continuous time forward rates: $f_{t,r} = e^{F_{t,r}} - 1$

Forward rate for the period $(t, t+r)$: $F_{t,r} = \frac{(t+r)Y_{t+r} - tY_t}{r} = \frac{1}{r}\log\left(\frac{P_t}{P_{t+r}}\right)$

Instantaneous forward rates: The instantaneous forward rate F_t is defined as: $F_t = \lim_{r\to 0} F_{t,r}$

$$F_t = -\frac{1}{P_t}\frac{d}{dt}P_t$$

$$P_t = e^{-\int_0^t F_s ds}$$

Yield curve: a graphical or tabular presentation of a collection of spot rates for various maturities n

Expectations Theory: The rate charged for a longer-term investments contains information about expected interest rates for future short-term investments

Liquidity Preference: Lenders prefer short-term bonds over long-term bonds because longer-term loans tie up their money for longer periods, reducing their flexibility to manage their capital

Market Segmentation: The term structure emerges from these different forces of supply and demand

Yields to maturity: The effective rate of interest at which the discounted value of the proceeds of a bond equal the price

Par yields: $1 = (yc_n)\left(v_{y_1} + v_{y_2}^2 + v_{y_3}^3 + \ldots + v_{y_n}^n\right) + 1v_{y_n}^n$

DURATION, CONVEXITY AND IMMUNISATION

Duration: $A = \sum_{k=1}^{n} C_{t_k} v_i^{t_k}$

Macaulay Duration: $\tau = \dfrac{\sum_{k=1}^{n} t_k C_{t_k} v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}} = -\dfrac{\frac{d}{d\delta}A}{A}$

Effective Duration: $v(i) = \dfrac{\sum_{k=1}^{n} C_{t_k} t_k v_i^{t_k+1}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}} = -\dfrac{\frac{d}{di}A}{A} = \dfrac{\tau}{1+i}$

Bond: $\tau = \dfrac{D(Ia)_{\overline{n|}} + Rnv^n}{Da_{\overline{n|}} + Rv^n}$

n-year zero coupon bond: $\tau = n$

Convexity: $c(i) = \dfrac{\sum_{t=1}^{n} C_{t_k} t_k (t_k+1) v_i^{t_k+2}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}} = \dfrac{\frac{d^2}{di^2}A}{A}$

Small change in interest rates ε:

$$\frac{A(i+\varepsilon) - A(i)}{A} = \frac{\partial A}{\partial i} \times \frac{1}{A} \times \varepsilon + 1/2 \times \frac{\partial^2 A}{\partial i^2} \times \frac{1}{A} \times \varepsilon^2 + \cdots$$
$$\approx -\varepsilon v(i) + 1/2\varepsilon^2 c(i)$$

Immunisation: Asset cashflows $\{A_{t_k}\}$, Liability cashflows $\{L_{t_k}\}$

Requirements for Redington Immunization: If same interest rate applies to all CF's

(i) PV(Assets) = PV(Liabilities) — (i) $V_A(i_0) = V_L(i_0)$

(ii) Volatility(Assets) = Volatility(Liabilities) — (ii) $v_A(i_0) = v_L(i_0)$

(iii) Convexity(Assets) > Convexity(Liabilities) — (iii) $c_A(i_0) > c_L(i_0)$

EQUATIONS OF VALUE

Net cashflow at time t_r: $c_{t_r} = b_{t_r} - a_{t_r}$ with outlays of amount a_{t_r} and receive payment b_{t_r}

Force of interest: $\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} = 0$

Yield equation: $\sum_{r=1}^{n} c_{t_r} v^{t_r} = 0$

Net rate of cashflow at time t: $\rho(t) = \rho_2(t) - \rho_1(t)$

Force of interest: $\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} + \int_0^\infty \rho(t) e^{-\delta t} dt = 0$ (discrete and continuous cashflows)

Yield equation: $\sum_{r=1}^{n} c_{t_r} (1+i)^{-t_r} + \int_0^\infty \rho(t)(1+i)^{-t} dt = 0$ (discrete and continuous cashflows)

Probability of cashflow:

Force of interest: $\sum_{r=1}^{n} c_{t_r} e^{-\delta t_r} e^{-\mu t_r} + \int_0^\infty \rho(t) e^{-\delta t} e^{-\mu t} dt = 0$

Yield equation: $\sum_{r=1}^{n} p_{t_r} c_{t_r} (1+i)^{-t_r} + \int_0^\infty p(t)\rho(t)(1+i)^{-t} dt = 0$

Higher discount rate: $\sum_{r=1}^{n} c_{t_r} e^{-\delta' t_r} + \int_0^\infty \rho(t) e^{-\delta' t} dt = 0$

where $\delta' = \delta + \mu$

LOANS

Loan Schedules: Repayment X_t, Outstanding Balance L_t at time t, Capital Repayment f_t, Interest b_t

Prospective $L_t = X_{t+1}v + X_{t+2}v^2 + X_{t+3}v^3 + \ldots + X_n v^{n-t}$

Retrospective $L_t = L_0(1+i)^t - \left(X_1(1+i)^{t-1} + X_2(1+i)^{t-2} + \ldots + X_{t-1}(1+i) + X_t\right)$

L_t, f_t, and b_t at time t

$L = L_0 = Xa_{\overline{n}|}$ $\qquad b_t = iL_{t-1}$ $\qquad f_n = L_{n-1}$

$L_{n-1} = X_n v$ $\qquad f_t = X_t - iL_{t-1}$ $\qquad L_t = L_{t-1}(1+i) - X_t$

Year $r \to r+1$	Loan outstanding at r	Instalment at $r+1$	Interest due at $r+1$	Capital repaid at $r+1$	Loan outstanding at $r+1$
$0 \to 1$	L_0	x_1	iL_0	$X_1 - iL_0$	$L_1 = L_0 - (X_1 - iL_0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t \to t+1$	L_t	X_{t+1}	iL_t	$X_{t+1} - iL_t$	$L_{t+1} = L_t - (X_{t+1} - iL_t)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1 \to n$	L_{n-1}	X_n	iL_{n-1}	$X_n - iL_{n-1}$	0

Repayment payable more frequently than annually

Prospectively: $L_t = X_{t+1/p}v^{1/p} + X_{t+2/p}v^{2/p} + \ldots + X_n v^{n-t}$

Retrospectively: $L_t = L_0(1+i)^t - \left(X_{\frac{1}{p}}(1+i)^{t-1/p} + X_{\frac{2}{p}}(1+i)^{t-2/p} + \ldots + X_{t-1/p}(1+i)^{1/p} + X_t\right)$

FIXED-INTEREST SECURITIES

Security has No tax: $P = Da^{(p)}_{\overline{n}|} + Rv^n$

Security with Income tax: $P' = (1-t_1)\,Da^{(p)}_{n} + Rv^n$ with income tax at rate t_1 on the coupons

Capital gains tax: The tax levied on the capital gain (the price paid for a bond is less than the redemption)

Capital gains test: There is a capital gain if $R > (1-t_1)\,Da^{(p)}_{\overline{n}|} + Rv^n \to i^{(p)} > (1-t_1)\dfrac{D}{R}$

$P'' = (1-t_1)\,Da^{(p)}_{\overline{n}|} + Rv^n - t_2\,(R - P'')\,v^n$

Callable Bonds To calculate appropriate price:

If Bond is sold at a **capital loss**, assume Earliest Redemption date

If Bond is sold at a **capital gain**, assume Latest Redemption date

Equity: $P = \sum_{t=1}^{\infty} D_t v_i^t$ and $P = \dfrac{D(1+g)}{i-g}$ with constant dividend growth rate of g

Property: $P = \sum_{k=1}^{\infty} \dfrac{1}{m} D_{k/m} v^{\frac{k}{m}}$

REAL RATES OF INTEREST

Cashflows $\{C_{t_1}, C_{t_2}, \ldots, C_{t_n}\}$ and associated inflation index values $\{Q(0), Q(t_1), Q(t_2), \ldots, Q(t_n)\}$

$$\rightarrow \sum_{k=1}^{n} \frac{C_{t_k}}{Q(t_k)} v_{i'}^{t_k} = 0$$

Real rate of interest: Real yield $= \dfrac{1 + \text{Annual rate of interest}}{1 + \text{Inflation rate}} - 1 \quad \rightarrow \quad i' = \dfrac{i - j}{1 + j}$

Payments related to the rate of inflation: $\sum_{k=1}^{n} c_{t_k} v_{i'}^{t_k} = 0$ where $C_t = c_t \dfrac{Q(t)}{Q(0)}$

Rate of escalation j: $c_t^e = (1+j)^t c_t$ and $\rho^e(t) = (1+j)^t \rho(t)$

Net present value:

$$NPV_j(i) = \sum c_t (1+j)^t (1+i)^{-t} + \int_0^\infty \rho(t)(1+j)^t (1+i)^{-t} dt$$
$$= \sum c_t (1+i_0)^{-t} + \int_0^\infty \rho(t)(1+i_0)^{-t} dt$$

Index-linked bonds: $P = \sum_{k=1}^{2n} \dfrac{D}{2} \dfrac{Q(k/2)}{Q(0)} v_i^{\frac{k}{2}} + R \dfrac{Q(n)}{Q(0)} v_i^n$

PROJECT APPRAISAL

Net cashflow c_t at time t $\quad$ $c_t =$ cash inflow at time t $-$ cash outflow at time t

Net rate of cashflow per unit time at time $\rho(t)$: $\quad \rho(t) = \rho_1(t) - \rho_2(t)$

where $\rho_1(t)$, $\rho_2(t)$ denote the rates of inflow and outflow at time t respectively

Net present value: $NPV(i) = \sum c_t (1+i)^{-t} + \int_0^T \rho(t)(1+i)^{-t} dt$

Internal rate of return: For the transaction is the interest rate at which the value of all cashflows out is equal to the value of cashflows in.

Discounted Payback Period: $A(t) = \sum_{s \le t} c_s (1+j_1)^{t-s} + \int_0^t \rho(s)(1+j_1)^{t-s} ds$

Accumulated value: $A(T) = \sum c_t (1+i)^{T-t} + \int_0^T \rho(t)(1+i)^{T-t} dt$

Accumulated profit: $P = A(t_1)(1+j_2)^{T-t_1} + \sum_{t > t_1} c_t (1+j_2)^{T-t} + \int_{t_1}^T \rho(t)(1+j_2)^{T-t} dt$

ACTUARIAL NOTATION

Survival probability: ${}_tp_x = \Pr(T_x > t)$

Mortality probability: ${}_tq_x = 1 - {}_tp_x = \Pr(T_x \le t)$

Formulas:

$${}_{t+s}p_x = {}_tp_x \; {}_sp_{x+t} = {}_sp_x \; {}_tp_{x+s}$$

$${}_{t+s}q_x = {}_tq_x + {}_tp_x \; {}_sq_{x+t}$$

$${}_{t|s}q_x = {}_tp_x \; {}_sq_{x+t} = {}_tp_x - {}_{t+s}p_x = {}_{t+s}q_x - {}_tq_x$$

LIFE TABLES

Number of lives: $l_x = l_\alpha \times {}_{x-\alpha}p_\alpha$ for $\alpha \le x \le \omega$ where ω is referred to as the limiting age of the table

Number of deaths: ${}_td_x = l_x - l_{x+t}$

Formulas:

$${}_tp_x = \frac{{}_{t+x-\alpha}p_\alpha}{{}_{x-\alpha}p_\alpha} = \frac{l_{x+t}}{l_\alpha} \times \frac{l_\alpha}{l_x} = \frac{l_{x+t}}{l_x}$$

$${}_tq_x = 1 - {}_tp_x = 1 - \frac{l_{x+t}}{l_x} = \frac{{}_td_x}{l_x} = \frac{l_x - l_{x+t}}{l_x}$$

$${}_{n|m}q_x = \frac{{}_md_{x+n}}{l_x} = \frac{l_{x+n} - l_{x+n+m}}{l_x}$$

$$\Pr[K_x = k] = {}_{k|}q_x$$

FORCE OF MORTALITY

Definition: $\mu_{x+t} = \mu = \text{constant}$

Formulas:

$${}_tp_x = e^{-\int_0^t \mu_{x+s}ds} = e^{-\int_x^{x+t}\mu_s ds}$$

$${}_tq_x = \int_0^t {}_sp_x\ \mu_{x+s}ds$$

$${}_{t-s}p_{x+s} = e^{-\int_s^t \mu_{x+r}ds} = e^{-u(t-s)}$$

SELECT SURVIVAL MODEL

k-year select period:

$q_{[x]+r} < q_{x+r}$ for $r < k$ $\qquad p_{[x]+r} > p_{x+r}$ for $r < k$

$q_{[x]+r} = q_{x+r}$ for $r \geq k$ $\qquad p_{[x]+r} = p_{x+r}$ for $r \geq k$

$$l_{[x]+t} = \frac{l_{[x]+t+1}}{(1-q_{[x]+t})} \qquad d_{[x]+r} = l_{[x]+r} - l_{[x]+r+1}$$

$${}_nq_{[x]+r} = \frac{l_{[x]+r} - l_{[x]+r+n}}{l_{[x]+r}} \qquad {}_{n|m}q_{[x]+r} = \frac{l_{[x]+r+n} - l_{[x]+r+n+m}}{l_{[x]+r}}$$

EXPECTED FUTURE LIFETIME

Expectations:

$$E[T_x] = \overset{o}{e}_x = \int_0^\infty t\ {}_tp_x\ \mu_{x+t}dt = \int_0^\infty {}_tp_x dt$$

$$E[K_x] = e_x = \sum_{k=1}^{\infty} k\ {}_kp_x\ q_{x+k} = \sum_{k=1}^{\infty} k\ {}_{k|}q_x = \sum_{k=1}^{\infty} {}_kp_x \approx \overset{o}{e} - 1/2$$

Second moments:

$$E\left[T_x^2\right] = \int_0^\infty t^2{}_tp_x\ \mu_{x+t}dt = \int_0^\infty 2t_tp_x dt$$

$$E\left[K_x^2\right] = \sum_{k=1}^{\infty} k^2{}_kp_x\ q_{x+k} = \sum_{k=1}^{\infty}(2k-1)\,{}_kp_x = 2\sum_{k=1}^{\infty} k\ {}_kp_x - e_x$$

Variance:

$$Var(T_x) = E\left[T_x^2\right] - E[T_x]^2$$

$$Var(K_x) = E\left[K_x^2\right] - E[K_x]^2$$

APPROXIMATIONS

UDD between integral ages: $l_{x+s} = l_x - sd_x \rightarrow {}_sq_x = \int_0^s q_x dt = sq_x \qquad {}_{t-s}q_{x+s} = \dfrac{(t-s)q_x}{1-sq_x}$

CFM between integral ages: $l_{x+s} = l_x \times (p_x)^s \rightarrow {}_sp_x = (p_x)^s \qquad {}_{t-s}p_{x+s} = (p_x)^{t-s}$

$${}_{t-s}p_{x+s} = \exp\left\{-\int_s^t \mu_{x+r}dr\right\} = e^{-\mu(t-s)}$$

These are for $0 \leq s,t \leq 1$ and $0 \leq s+t \leq 1$

ACTUARIAL FUNCTIONS

Assurance (Discrete)

Whole Life Assurance: $A_x = E\left[v^{K_x+1}\right] = \sum_{k=0}^{\infty} v^{k+1}{}_{k|}q_x = \sum_{k=0}^{\infty} v^{k+1}{}_kp_x\ q_{x+k}$

Term Life Assurance: $A^{1}_{x:\overline{n}|} = E[F] = \sum_{k=0}^{n-1} v^{k+1}{}_{k|}q_x = \sum_{k=0}^{n-1} v^{k+1}{}_kp_x\ q_{x+k}$

Pure Endowment: $A_{x:\overline{n}|}^{\ \ 1} = E[G] = v^n{}_np_x$

Endowment Assurance: $A_{x:\overline{n}|} = E[H] = \sum_{k=0}^{n-1} v^{k+1}{}_{k|}q_x + v^n{}_np_x = \sum_{k=0}^{n-1} v^{k+1}{}_kp_x\ q_{x+k} + v^n{}_np_x$

$$A_{[x]} = \sum_{k=0}^{\infty} v^{k+1}{}_{k|}q_{[x]}$$

Assurance (Continuous)

Whole life Assurance: $\bar{A}_x = \mathrm{E}\left[v^{\mathrm{T}_x}\right] = \int_0^\infty v^t {}_tp_x\ \mu_{x+t}dt$

$^2\bar{A}_x = \int_0^\infty \left(v^t\right)^2 {}_tp_x\ \mu_{x+t}dt$

Term Life Assurance: $\bar{A}_{\overset{1}{x}:\overline{n}|} = E[\bar{\mathrm{F}}] = \int_0^n v^t {}_tp_x\ \mu_{x+t}dt$

$^2\bar{A}_{\overset{1}{x}:\overline{n}|} = \int_0^n \left(v^t\right)^2 {}_tp_x\ \mu_{x+t}dt$

Endowment Assurance: $\bar{A}_{x:\overline{n}|} = E[\bar{\mathrm{H}}] = \int_0^n v^t {}_tp_x\ \mu_{x+t}dt + v^n {}_np_x$

$^2\bar{A}_{x:\overline{n}|} = \int_0^n \left(v^t\right)^2 {}_tp_x\ \mu_{x+t}dt + \left(v^n\right)^2 {}_np_x$

Assurance (mthly): $A^{(m)}_{\overset{1}{x}:\overline{n}|} = \sum_{k=0}^{nm-1} v^{k/m+1/m} {}_{k/m}p_x\ {}_{1/m}q_{x+k/m}$

Relations:

$\bar{A}_x = \bar{A}_{\overset{1}{x}:\overline{n}|} + {}_{n|}\bar{A}_x = \bar{A}_{\overset{1}{x}:\overline{n}|} + v^n {}_np_x\ \bar{A}_{x+n}$

$A_x = A_{\overset{1}{x}:\overline{n}|} + v^n {}_np_x\ A_{x+n}$

$\bar{A}_{x:\overline{n}|} = \bar{A}_{\overset{1}{x}:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}} = \bar{A}_{\overset{1}{x}:\overline{n}|} + v^n {}_np_x$

$A_{x:\overline{n}|} = A_{\overset{1}{x}:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}} = A_{\overset{1}{x}:\overline{n}|} + v^n {}_np_x$

${}_{n|}\bar{A}_x = v^n {}_np_x\ \bar{A}_{x+n}$

${}_{n|}A_x = A_x - A_{\overset{1}{x}:\overline{n}|} = v^n {}_np_x A_{x+n}$

VARIANCE OF PRESENT VALUES

Policy	Death benefit paid at the moment of death	Death benefit paid at the end of the year of death
Whole life insurance	$\mathrm{Var}\left[v^{\mathrm{T}_x}\right] = {}^2\bar{A}_x - \left(\bar{A}_x\right)^2$	$\mathrm{Var}\left[v^{\mathrm{K}_x+1}\right] = \sum_{k=0}^{\infty}\left(v^{k+1}\right)^2 {}_{k\|}q_x - \left(A_x\right)^2$
n-year term insurance	$\mathrm{Var}[\bar{\mathrm{F}}] = {}^2\bar{A}_{\overset{1}{x}:\overline{n}\|} - \left(\bar{A}_{\overset{1}{x}:\overline{n}\|}\right)^2$	$\mathrm{Var}[\mathrm{F}] = {}^2A_{\overset{1}{x}:\overline{n}\|} - \left(A_{\overset{1}{x}:\overline{n}\|}\right)^2$
n-year endowment insurance	$\mathrm{Var}[\bar{\mathrm{H}}] = {}^2\bar{A}_{x:\overline{n}\|} - \left(\bar{A}_{x:\overline{n}\|}\right)^2$	$\mathrm{Var}[\mathrm{H}] = {}^2A_{x:\overline{n}\|} - \left(A_{x:\overline{n}\|}\right)^2$
n-year pure endowment		$\mathrm{Var}[\mathrm{G}] = {}^2A_{x:\overset{1}{\overline{n}\|}} - \left(A_{x:\overset{1}{\overline{n}\|}}\right)^2$
n-year deferred life insurance	$\mathrm{Var}[\bar{\mathrm{J}}] = {}^2_{n\|}\bar{A}_x - \left({}_{n\|}\bar{A}_x\right)^2$	$\mathrm{Var}[\mathrm{J}] = {}^2_{n\|}A_x - \left({}_{n\|}A_x\right)^2$

APPROXIMATIONS

UDD between integral ages:

$\bar{A}_x = \frac{i}{\delta}A_x$ $\qquad \bar{A}_{\overset{1}{x}:\overline{n}|} = \frac{i}{\delta}A_{\overset{1}{x}:\overline{n}|}$ $\qquad \bar{A}_{x:\overline{n}|} \neq \frac{i}{\delta}A_{x:\overline{n}|}.$

$A^{(m)}_x = \frac{i}{i^{(m)}}A_x$ $\qquad {}^2\bar{A}_x = \frac{2i+i^2}{2\delta}\ {}^2A_x$

Claims acceleration approach:

$\bar{A}_x = (1+i)^{1/2} A_x$ $\qquad \bar{A}_{\overset{1}{x}:\overline{n}|} = (1+i)^{1/2} A_{\overset{1}{x}:\overline{n}|}$

$A^{(m)}_x = (1+i)^{\frac{m-1}{2m}} A_x$ $\qquad {}^2\bar{A}_x = (1+i)\ {}^2A_x$

$\bar{A}_{\overset{1}{x}:\overline{n}|} = (1+i)^{1/2} A_{\overset{1}{x}:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$ $\qquad (I\bar{A})_x \cong (1+i)^{1/2}(IA)_x$

ACTUARIAL FUNCTIONS

Annuity (Discrete)

Whole Life Annuity

Paid in advance: $\ddot{a}_x = E\left[\ddot{a}_{\overline{K_x+1}|}\right] = \sum_{j=0}^{\infty} {}_jp_x v^j$

Paid in arrears: $a_x = E\left[a_{\overline{K_x}|}\right] = \sum_{k=0}^{\infty} a_{\overline{k}|}\,{}_{k|}q_x = \sum_{j=1}^{\infty} {}_jp_x v^j$

Temporary Annuity

Paid in advance: $\ddot{a}_{x:\overline{n}|} = E\left[\ddot{a}_{\overline{\text{MIN}[K_x+1,n]}|}\right] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{k+1}|}\,{}_{k|}q_x + \ddot{a}_{\overline{n}|}\,{}_np_x = \sum_{j=0}^{n-1} {}_jp_x v^j$

Paid in arrears: $a_{x:\overline{n}|} = E\left[a_{\overline{\text{MIN}[K_x,n]}|}\right] = \sum_{k=1}^{n-1} a_{\overline{k}|}\,{}_{k|}q_x + a_{\overline{n}|}\,{}_np_x = \sum_{j=1}^{n} {}_jp_x v^j$

Deferred Annuity: ${}_{n|}\ddot{a}_x = \sum_{k=n}^{\infty} v^k\,{}_kp_x$

Guaranteed annuity

Paid in advance: $\ddot{a}_{\overline{x:\overline{n}|}} = E\left[\ddot{a}_{\overline{\text{MAX}[K_x+1,n]}|}\right] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{n}|}\,{}_{k|}q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1}|}\,{}_{k|}q_x = \ddot{a}_{\overline{n}|} + \sum_{j=n}^{\infty} {}_jp_x v^j$

$a_{\overline{x:\overline{n}|}} = E\left[a_{\overline{\text{MAX}[K_x,n]}|}\right] = \sum_{k=0}^{n-1} a_{\overline{n}|}\,{}_{k|}q_x + \sum_{k=n}^{\infty} a_{\overline{k}|}\,{}_{k|}q_x = a_{\overline{n}|} + \sum_{j=n+1}^{\infty} {}_jp_x v^j$

$a_x^{(m)} = \frac{1}{m}\sum_{t=1}^{\infty} \frac{v^{t/m} l_{x+t/m}}{l_x}$ $\quad \ddot{a}_{[x]} = \sum_{0}^{\infty} {}_kp_{[x]} \cdot v^k$

Annuity (Continuous)

Whole Life Annuity: $\bar{a}_x = E\left[\bar{a}_{\overline{T_x}|}\right] = \int_0^{\infty} \bar{a}_{\overline{t}|}\,{}_tp_x\ \mu_{x+t}dt = \int_0^{\infty} v^t\,{}_tp_x dt$

Temporary Annuity: $\bar{a}_{x:\overline{n}|} = E\left[\bar{a}_{\overline{\text{MIN}[T_x,n]}|}\right] = \int_0^{n} \bar{a}_{\overline{t}|}\,{}_tp_x\ \mu_{x+t}dt + \bar{a}_{\overline{n}|}\,{}_np_x$

Relations:

$\bar{a}_x = \bar{a}_{x:\overline{n}|} + {}_{n|}\bar{a}_x$ $\quad \ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_{n|}\ddot{a}_x$ $\quad a_x = a_{x:\overline{n}|} + {}_{n|}a_x$

${}_{n|}\bar{a}_x = v^n\,{}_np_x\ \bar{a}_{x+n}$ $\quad {}_{n|}\ddot{a}_x = v^n\,{}_np_x\ \ddot{a}_{x+n}$ $\quad {}_{n|}a_x = v^n\,{}_np_x\ a_{x+n}$

$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + {}_{n|}\bar{a}_x$ $\quad \ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_{n|}\ddot{a}_x$ $\quad a_{\overline{x:\overline{n}|}} = a_{\overline{n}|} + {}_{n|}a_x$

$\ddot{a}_x = 1 + a_x$ $\quad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|} = 1 + a_{x:\overline{n}|} - v^n\,{}_np_x$

$a_x = vp_x\ddot{a}_{x+1}$ $\quad a_{x:\overline{n}|} = vp_x\ddot{a}_{x+1:\overline{n}|}$ $\quad {}_{\frac{t}{m}|}\ddot{a}_x \cong \ddot{a}_x - \frac{t}{m}$

$\bar{a}_x \cong \ddot{a}_x - 1/2$ $\quad \bar{a}_{x:\overline{n}|} \cong \ddot{a}_{x:\overline{n}|} - 1/2\left(1 - v^n\,{}_np_x\right)$ $\quad (I\bar{a})_x \approx (I\ddot{a})_x - \frac{1}{2}\ddot{a}_x$

$\ddot{a}_x^{(m)} \cong \ddot{a}_x - \frac{(m-1)}{2m}$ $\quad \ddot{a}_x^{(m)} = \frac{1}{m} + a_x^{(m)}$ $\quad a_x^{(m)} \cong a_x + \frac{m-1}{2m}$

Assurance to annuity:

$\ddot{a}_x = \frac{1 - A_x}{d}$ $\quad \bar{a}_x = \frac{1 - \bar{A}_x}{\delta}$ $\quad \ddot{a}_{[x]} = \frac{1 - A_{[x]}}{d}$

$\ddot{a}_{x:\overline{n}|}^{(m)} = \frac{1 - A_{x:\overline{n}|}^{(m)}}{d^{(m)}}$ $\quad \ddot{a}_{x:\overline{n}|} = \frac{1 - A_{x:\overline{n}|}}{d}$ $\quad \bar{a}_{x:\overline{n}|} = \frac{1 - \bar{A}_{x:\overline{n}|}}{\delta}$

$\ddot{a}_{[x]:\overline{n}|} = \frac{1 - A_{[x]:\overline{n}|}}{d}$

VARIANCE OF PRESENT VALUES

Policy	Paid continuously	Paid in arrears	Paid in advance
Whole life annuity	$\frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}$	$\frac{{}^2A_x - (A_x)^2}{d^2}$	$\frac{{}^2A_x - (A_x)^2}{d^2}$
n-year term annuity	$\frac{{}^2\bar{A}_{x:\overline{n}\vert} - (\bar{A}_{x:\overline{n}\vert})^2}{\delta^2}$	$\frac{{}^2A_{x:\overline{n+1}\vert} - \left(A_{x:\overline{n+1}\vert}\right)^2}{d^2}$	$\frac{{}^2A_{x:\overline{n}\vert} - (A_{x:\overline{n}\vert})^2}{d^2}$
Guaranteed annuity		$\frac{v^{2n}{}_nq_x + {}_{n\vert}^{2}A_x - (v^n{}_nq_x + {}_{n\vert}A_x)^2}{d^2}$	$\frac{v^{2n}{}_nq_x + {}_{n\vert}^{2}A_x - (v^n{}_nq_x + {}_{n\vert}A_x)^2}{d^2}$

RETROSPECTIVE ACCUMULATIONS

Retrospective accumulation: $\lim_{L\to\infty} \frac{\mathbf{F}_n(L)}{\mathrm{L}_n} = \frac{E[\mathbf{F}_n(1)]}{{}_np_x}$

Pure Endowment: $E[\mathbf{F}_n(1)] = {}_np_x \quad \rightarrow \quad \lim_{L\to\infty} \frac{\mathbf{F}_n(L)}{\mathrm{L}_n} = \frac{{}_np_x}{{}_np_x} = 1$

Term Assurance: $E[\mathbf{F}_n(1)] = (1+i)^n A^1_{x:\overline{n}\vert} \quad \rightarrow \quad \lim_{L\to\infty} \frac{\mathbf{F}_n(L)}{\mathrm{L}_n} = \frac{(1+i)^n A^1_{x:\overline{n}\vert}}{{}_np_x}$

Temporary Annuity-due: $E[\mathbf{F}_n(1)] = ((1+i)^n \ddot{a}_{x:\overline{n}\vert} \quad \rightarrow \quad \lim_{L\to\infty} \frac{\mathbf{F}_n(L)}{\mathrm{L}_n} = \frac{(1+i)^n \ddot{a}_{x:\overline{n}\vert}}{{}_np_x}$

Accumulation value: $\ddot{s}_{x:\overline{n}\vert} = \frac{(1+i)^n \ddot{a}_{x:\overline{n}\vert}}{{}_np_x}$

VALUING VARIABLE BENEFITS AND ANNUITIES

Expected present value: $Y_x v \frac{d_x}{l_x} + Y_{x+1} v^2 \frac{d_{x+1}}{l_x} + \ldots + Y_{x+t} v^{t+1} \frac{d_{x+t}}{l_x} + \ldots$

with payment Y_x when death occurs in the year of age $(x, x+1)$

EPV of an annuity: $F_{x+1} v \frac{l_{x+1}}{l_x} + F_{x+2} v^2 \frac{l_{x+2}}{l_x} + \ldots + F_{x+t} v^t \frac{l_{x+t}}{l_x} + \ldots$

with amount F_{x+t} payable on survival to age $x+t$

Payments varying at a constant compound rate

$\frac{1}{1+b} A^j_x$ with payment $(1+b)^k$ when death occurs in the year of age $(x+k, x+k+1)$

where $j = \frac{(1+i)}{(1+b)} - 1$

Immediate annuity: a^j_x with amount $(1+c)^k$ payable on survival to age $x+k$

where $j = \frac{(1+i)}{(1+c)} - 1$

Payments varying by constant monetary amount

Payment $k+1$ when death occurs in the year of age $(x+K, x+K+1)$

Increasing whole life assurance: $(IA)_x = \sum_{k=0}^{\infty} (k+1) v^{k+1} {}_{k\vert}q_x$

Increasing temporary assurance: $(IA)^1_{x:\overline{n}\vert} = (IA)_x - v^n \frac{l_{x+n}}{l_x} [nA_{x+n} + (IA)_{x+n}]$

Increasing endowment assurance: $(IA)_{x:\overline{n}\vert} = (IA)^1_{x:\overline{n}\vert} + nA_{x:\overline{n}\vert}^{\ \ 1} = (IA)^1_{x:\overline{n}\vert} + n\frac{D_{x+n}}{D_x}$

Decreasing temporary assurance: $(n+1)A^1_{x:\overline{n}\vert} - (IA)^1_{x:\overline{n}\vert}$

Increasing whole life annuity: $(Ia)_x = \sum_{k=1}^{\infty} k v^k {}_kp_x$

Increasing whole life annuity due: $(I\ddot{a})_x = \sum_{k=0}^{\infty}(k+1)v^k\,{}_kp_x$

$(I+\ddot{a})_x = (Ia)_x + \ddot{a}_x$

Increasing temporary annuity: $(I\ddot{a})_{x:\overline{n}|} = (I\ddot{a})_x - v^n \frac{l_{x+n}}{l_x}\left[n\ddot{a}_{x+n} + (I\ddot{a})_{x+n}\right]$

Decreasing temporary annuity: $(n+1)\ddot{a}_{x:\overline{n}|} - (I\ddot{a})_{x:\overline{n}|}$

UNIT-LINKED AND WITH-PROFITS CONTRACTS

Unit-linked contracts:

Unit-linked assurances: Have benefits which are directly linked to the value of the underlying investments
Each policyholder receives the value of the units allocated to the policy

Guaranteed benefits: (1) on death during the policy term, the higher of a fixed sum assured or the value of units might be paid

(2) on survival to the maturity date of the policy, a minimum guaranteed sum assured, or a minimum average unit growth rate, may be applied

Conventional with-profits contracts

without profits basis: both the premiums and benefits under the policy are usually fixed and guaranteed at the date of issue

with-profits basis: the premiums and/or the benefits can be varied to give an additional benefit to the policyholder in respect of any emerging surplus of assets over liabilities following a valuation

Simple bonus: the bonus rate is applied to the basis sum assured

Compound bonus: the bonus rate is applied to the basic sum assured and bonuses added in the past

Super-compound bonus: two compound bonus rates are declared every year, one applying to the basic sum assured, and one to the bonuses added to the policy in the past

Accumulating with-profits contracts:

Accumulating fund at time t: $F_t = (F_{t-1} + P)(1 + b_t)$ with annual premiums of P and annual bonus interest b_t

$F_t = (F_{t-1} + P)(1 + g)(1 + b_t)$ including a guaranteed bonus interest rate of g per annum

Contractual benefit: $B_t = F_t + T_t$ where T is the amount of terminal bonus payable on a claim at time t

Unitised with profits (UWP): Method (1) the unit price allows for guaranteed bonus interest increases only; the discretionary bonus is credited to the policy by awarding additional (bonus) units from time to time.

Method (2) the unit price allows for both guaranteed and bonus interest increases.

Benefit: $\max[S, B_t] = \max[S, F_t + T_t]$

FUNCTIONS INVOLVING TWO LIVES

Joint life functions

Random variable: $T_{xy} = \min\{T_x, T_y\}$

CDF of T_{xy}: $F_{T_{xy}}(t) = P[T_{xy} \le t] = 1 - {}_tp_{xy} = 1 - P[T_x > t]\,P[Y_y > t] = 1 - {}_tp_x\,{}_tp_y$

Density function of T_{xy}: $f_{T_{xy}}(t) = {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right) = {}_tp_{xy}\ \mu_{x+t:y+t}$

Joint life table: ${}_tp_{xy} = \dfrac{\ell_{x+t}}{\ell_x} \cdot \dfrac{\ell_{y+t}}{\ell_y} = \dfrac{\ell_{x+t:y+t}}{\ell_{xy}}$ where $\ell_{xy} = \ell_x \ell_y$

$d_{xy} = \ell_{xy} - \ell_{x+1:y+1}$ $\quad q_{xy} = \dfrac{d_{xy}}{\ell_{xy}}$ $\quad \mu_{x+t:y+t} = -\dfrac{1}{\ell_{x+t:y+t}}\dfrac{d}{dt}\ell_{x+t:y+t} = \mu_{x+t} + \mu_{y+t}$

Probability function of K_{xy}: $P[K_{xy} = k] = P[k \le T_{xy} < k+1] = {}_{k|}q_{xy}$

Last survivor function

Random variable: $T_{\overline{xy}} = \max\{T_x, T_y\}$

CDF of T_{xy}: $F_{T_{\overline{xy}}}(t) = P[T_{\overline{xy}} \le t] = P[T_x \le t]\,P[T_y \le t] = (1 - {}_tp_x)(1 - {}_tp_y)$

$= F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t)$

Density function of $T_{\overline{xy}}$: $f_{T_{\overline{xy}}}(t) = {}_tp_x\ \mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_x\,{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t)$

Probability function of $K_{\overline{xy}}$: $P[K_{\overline{xy}} = k] = P[k \le T_{\overline{xy}} < k+1] = {}_{k|}q_x + {}_{k|}q_y - {}_{k|}q_{xy}$

Relationship: $T_{xy} + T_{\overline{xy}} = \min\{T_x, T_y\} + \max\{T_x, T_y\} = T_x + T_y$

$K_{xy} + K_{\overline{xy}} = \min\{K_x, K_y\} + \max\{K_x, K_y\} = K_x + K_y$

Assurance functions: Status u could be any joint lifetime or last survivor status, e.g. $xy, \overline{xy}$

$\bar{A}_u = E\left[\bar{Z}_u\right] = \int_{t=0}^{t=\infty} v^t f_{T_u}(t)dt$

$\text{Var}\left(\bar{Z}_u\right) = {}^2\bar{A}_u - \left(\bar{A}_u\right)^2$

Continuous joint life: $\bar{A}_{xy} = \int_{t=0}^{t=\infty} v^t\,{}_tp_{xy}\ \mu_{x+t:y+t}dt$ $\qquad {}^2\bar{A}_{xy} - \left(\bar{A}_{xy}\right)^2$

Continuous last survivor: $\bar{A}_{\overline{xy}} = \int_{t=0}^{t=\infty} v^t\left({}_tp_x\ \mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\ \mu_{x+t:y+t}\right)dt = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$

${}^2\bar{A}_{\overline{xy}} - \left(\bar{A}_{\overline{xy}}\right)^2 = \left({}^2\bar{A}_x + {}^2\bar{A}_y - {}^2\bar{A}_{xy}\right) - \left(\bar{A}_x + \bar{A}_y - \bar{A}_{xy}\right)^2$

Discrete joint life: $A_{xy} = \sum_{t=0}^{\infty} v^{t+1}\,{}_{t|}q_{xy}$ $\qquad {}^2A_{xy} - \left(A_{xy}\right)^2$

Discrete last survivor: $A_{\overline{xy}} = A_x + A_y - A_{xy}$

${}^2A_{\overline{xy}} - \left(A_{\overline{xy}}\right)^2 = \left({}^2A_x + {}^2A_y - {}^2A_{xy}\right) - \left(A_x + A_y - A_{xy}\right)^2$

Annuity functions:

Continuous Annuity: $E\left[\bar{a}_{\overline{T_u}|}\right] = \bar{a}_u = \int_{t=0}^{t=\infty} \bar{a}_{\overline{t}|} f_{T_u}(t)dt = \dfrac{1 - \bar{A}_u}{\delta}$

$\text{Var}\left(\bar{a}_{\overline{T_u}|}\right) = \text{Var}\left(\dfrac{1 - v^{T_u}}{\delta}\right) = \dfrac{1}{\delta^2}\left\{{}^2\bar{A}_u - \left(\bar{A}_u\right)^2\right\}$

Discrete annuity due: $\ddot{a}_u = \dfrac{1 - A_u}{d}$ $\qquad \dfrac{1}{d^2}\left\{{}^2A_u - \left(A_u\right)^2\right\}$

Discrete immediate annuity: $a_u = \dfrac{(1-d) - A_u}{d}$ $\qquad \dfrac{1}{d^2}\left\{{}^2A_u - \left(A_u\right)^2\right\}$

Relations: $a_{\overline{xy}}^{(m)} = a_x^{(m)} + a_y^{(m)} - a_{xy}^{(m)} \cong a_x + a_y - a_{xy} + \frac{m-1}{2m}$

$A_{xy} = 1 - d\ddot{a}_{xy}$ $\qquad \bar{A}_{xy} = 1 - \delta\bar{a}_{xy}$

$A_{\overline{xy}} = 1 - d\ddot{a}_{\overline{xy}}$ $\qquad \bar{A}_{\overline{xy}} = 1 - \delta\bar{a}_{\overline{xy}}$

CONTINGENT

Contingent probabilities of death

Events: $\overset{1}{x}y$, the event that (x) is the first to die of two lives (x) and (y)

$\overset{2}{x}y$, the event that (x) is the second to die of two lives (x) and (y).

Probability:

$${}_nq_{\overset{1}{x}y} = \int_{t=0}^{t=n} {}_tp_x\mu_{x+t}\left\{\int_{s=t}^{s=\infty} {}_sp_y\mu_{y+s}\cdot ds\right\}dt = \int_{t=0}^{t=n} {}_tp_{xy}\mu_{x+t}dt$$

$${}_nq_{\overset{2}{x}y} = \int_{t=0}^{t=n} (1-{}_tp_y)_t\,p_x\mu_{x+t}\cdot dt = {}_nq_x - {}_nq_{\overset{1}{x}y}$$

Relations:

$${}_nq_x = {}_nq_{\overset{1}{x}y} + {}_nq_{\overset{2}{x}y} \qquad {}_nq_{\overset{2}{x}x} = {}_nq_{\overset{1}{x}x} - {}_np_{x\,n}q_y$$

$${}_nq_{\overset{1}{x}x} = {}^1\!/{}_2\,{}_nq_{xx} \qquad {}_\infty q_{\overset{1}{x}x} = {}_\infty q_{\overset{2}{x}x} = {}^1\!/{}_2$$

Contingent assurance:

$$\bar{A}_{\overset{1}{x}y} = E[\bar{Z}] = \int_{t=0}^{t=\infty} v^t{}_tp_{xy}\,\mu_{x+t}dt \qquad \text{Var}(\bar{Z}) = {}^2\bar{A}_{\overset{1}{x}y} - \left(\bar{A}_{\overset{1}{x}y}\right)^2$$

$$\bar{A}_{\overset{1}{x}y} = \sum_{t=0}^{\infty} v^{t+1}{}_tp_{xy}q_{\overset{1}{x+t}:y+t}$$

Relations:

$$\bar{A}_{xy} = \bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}} \qquad \bar{A}_x = \bar{A}_{\overset{1}{x}y} + \bar{A}_{\overset{2}{x}y}$$

$$\bar{A}_{\overset{1}{x}x} = {}^1\!/{}_2\bar{A}_{xx} \qquad \bar{A}_{\overset{2}{x}x} = {}^1\!/{}_2\bar{A}_{\overline{xx}}$$

$$A_{\overset{1}{x}y} \approx (1+i)^{-1/2}\bar{A}_{\overset{1}{x}y}$$

Reversionary annuity:

$$\bar{a}_{x|y} = E[\bar{Z}] = \bar{a}_y - \bar{a}_{xy} = \frac{\bar{A}_{xy}-\bar{A}_y}{\delta} = \int_{t=0}^{t=\infty} v^t\bar{a}_{y+t\,t}p_{xy}\mu_{x+t}dt$$

$$a_{x|y} = E[Z] = a_y - a_{xy} = \frac{A_{xy}-A_y}{d}$$

$$a_{x|y}^{(m)} = a_y^{(m)} - a_{xy}^{(m)}$$

JOINT LIFE FUNCTIONS DEPENDENT ON TERM

Assurances functions:

$$\bar{A}^{\;1}_{\overline{xy}:\overline{n}|} = \int_{t=0}^{t=n} v^t{}_tp_{xy}\mu_{x+t:y+t}dt$$

$$\bar{A}_{\overset{1}{x}y:\overline{n}|} = \int_{t=0}^{t=n} v^t{}_tp_{xy}\mu_{x+t}dt$$

$$\bar{A}_{xy:\overset{\;1}{\overline{n}|}} = {}_np_{xy}v^n$$

Relations:

$$\bar{A}_{xy:\overline{n}|} = \bar{A}^{\;1}_{\overline{xy}:\overline{n}|} + \bar{A}_{xy:\overset{\;1}{\overline{n}|}}$$

$$\bar{A}_{\overline{xy}:\overline{n}|} = \bar{A}_{x:\overline{n}|} + \bar{A}_{y:\overline{n}|} - \bar{A}_{xy:\overline{n}|}$$

$$\bar{A}^{\;1}_{\overline{xy}:\overline{n}|} = \bar{A}^{\;1}_{x:\overline{n}|} + \bar{A}^{\;1}_{y:\overline{n}|} - \bar{A}^{\;1}_{\overline{xy}:\overline{n}|}$$

Annuity functions:

$$\bar{a}_{xy:\overline{n}|} = \int_{t=0}^{t=n} v^t_t p_{xy}dt$$

$$\bar{a}_{\overline{n}||y} = {}_{n|}\bar{a}_y = \bar{a}_y - \bar{a}_{y:\overline{n}|}$$

$$\bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|} = \bar{a}_y - \bar{a}_{xy} - v^n{}_np_{xy}\left(\bar{a}_{y+n} - \bar{a}_{x+n:y+n}\right)$$

$$\bar{a}_{y:\overline{n}|} + v^n{}_np_y\bar{a}_{x:y+n} - \bar{a}_{xy}$$

Relations:

$$\bar{a}_{\overline{xy}:\overline{n}|} = \bar{a}_{x:\overline{n}|} + \bar{a}_{y:\overline{n}|} - \bar{a}_{xy:\overline{n}|}$$

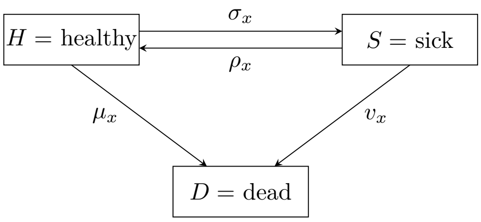
$$a^{(m)}_{\overline{xy}:\overline{n}|} = a^{(m)}_{x:\overline{n}|} + a^{(m)}_{y:\overline{n}|} - a^{(m)}_{xy:\overline{n}|}$$

$$a^{(m)}_{xy:\overline{n}|} \cong a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(1 - v^n\frac{\ell_{x+n}\ell_{y+n}}{\ell_x\ell_y}\right)$$

$$a^{(m)}_{y:\overline{n}|} - a^{(m)}_{xy:\overline{n}|} \cong a_{y:\overline{n}|} - a_{xy:\overline{n}|} + \frac{m-1}{2m}\left(v^n\frac{\ell_{x+n}\ell_{y+n}}{\ell_x\ell_y} - v^n\frac{\ell_{y+n}}{\ell_y}\right)$$

MULTIPLE TRANSITIONS

Multiple state model:



Transition probability:

${}_tp_x^{ij}$ is the probability that a life aged x who is currently in state i will be in state j at time t.

${}_tp_x^{ii}$ is the probability that a life aged x who is currently in state i will be in state i at time t.

${}_tp_x^{\overline{ii}}$ is the probability that a life aged x who is currently in state i continuously will be in state i until time t.

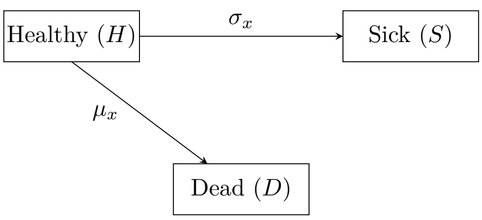
Forces of transition: $${}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j\neq i} \mu_{x+s}^{ij} ds\right)$$

EPV of lump sum $\int_0^\infty e^{-\delta t}\left({}_tp_x^{HH}\mu_{x+t} + {}_tp_x^{HS}v_{x+t}\right)dt$

EPV of annuity $\int_0^\infty e^{-\delta t}{}_tp_x^{HS}dt$

EPV of premium $\int_0^\infty e^{-\delta t}{}_tp_x^{HH}dt$

Multiple decrement model:



Probability:

$(aq)_x^s = {}_1p_x^{HS}$ $\quad (aq)_x^d = {}_1p_x^{HD}$ $\quad (ap)_x = {}_1p_x^{HH}$

$(ap)_x + (aq)_x^s + (aq)_x^d = 1$ $\quad (aq)_x = (aq)_x^s + (aq)_x^d$

$(ap)_x + (aq)_x = 1$

Transition probability:

${}_t(ap)_x = e^{-(\mu+\sigma)t}$ $\quad (aq)_x^s = \dfrac{\sigma}{\mu+\sigma}\left(1-e^{-(\mu+\sigma)}\right) = \dfrac{\sigma}{\mu+\sigma}(aq)_x$

$(aq)_x^d = \dfrac{\mu}{\mu+\sigma}\left(1-e^{-(\mu+\sigma)}\right)$

$q_x^s = 1-e^{-\sigma} \rightarrow \sigma = -\ln(1-q_x^s)$ $\quad q_x^d = 1-e^{-\mu}$

Multiple decrement table:

$(aq)_x^k = \dfrac{(ad)_x^k}{(al)_x}$ $\quad {}_n(aq)_x^k = \dfrac{(ad)_x^k + (ad)_{x+1}^k + \ldots + (ad)_{x+n-1}^k}{(al)_x}$

$(ap)_x = \dfrac{(al)_{x+1}}{(al)_x}$ $\quad {}_n(ap)_x = \dfrac{(al)_{x+n}}{(al)_x}$ $\quad {}_{n|}(aq)_x^k = \dfrac{(ad)_{x+n}^k}{(al)_x}$

$$(al)_{x+1} = (al)_x - \sum_k (ad)_x^k$$

$$\sigma = \frac{(aq)_x^s}{(aq)_x}(\mu+\sigma) = \frac{(aq)_x^s}{(aq)_x}(-\ln(ap)_x)$$

Associated single decrement table:

${}_tp_x^j = \exp\left\{\left(-\int_0^t \mu_{x+s}^j ds\right)\right\}$ $\quad {}_tq_x^j = \int_0^t {}_sp_x^j\mu_{x+s}^j ds$

$(a\mu)_x^j = \mu_x^j$ for all j and all x

GROSS PREMIUMS

Equivalence principle: E [Net future loss]$= EPV(\text{Benefits}) - EPV(\text{Premiums}) = 0$

E [Gross future loss]$= EPV(\text{Benefits}) + EPV(\text{Expenses}) - EPV(\text{Premiums}) = 0$

L: present value of the future outgo − present value of the future income

I: initial expenses in excess of those occurring regularly each year

e: level annual expenses

f: additional expenses incurred when the contract terminates

Annual premium contracts

whole life assurance	**Discrete**	$SA_x + I + e\ddot{a}_x + fA_x = G\ddot{a}_x$
	Continuous	$S\bar{A}_x + I + e\bar{a}_x + f\bar{A}_x = G\bar{a}_x$
Endowment assurance	**Discrete**	$SA_{x:\overline{n}\vert} + I + e\ddot{a}_x + fA_{x:\overline{n}\vert} = G\ddot{a}_{x:\overline{n}\vert}$
	Continuous	$S\bar{A}_{x:\overline{n}\vert} + I + e\bar{a}_{x:\overline{n}\vert} + f\bar{A}_x = G\bar{a}_{x:\overline{n}\vert}$

Conventional with-profits contracts: $S\frac{1}{1+b}A_x^j + I + e\ddot{a}_x + fA_x = G\ddot{a}_x$

Premiums payable m times per year: $S\bar{A}_x + I + e\ddot{a}_x^{(m)} + f\bar{A}_x = G\ddot{a}_x^{(m)}$

RESERVES

Prospective reserves

Net premium reserve: ${}_tV^n = E\left[{}_tL \mid T_x \geq t\right] = EPV_t\ (\text{Benefits}) - EPV_t\ (\text{Net Premiums})$

Gross premium reserve: ${}_tV^g = E\left[{}_tL^g \mid T_x \geq t\right]$

$= EPV_t\ (\text{Benefits}) + EPV_t\ (\text{Expenses}) - EPV_t\ (\text{Gross Premiums})$

whole life assurance	**Discrete**	$SA_{x+t} + e\ddot{a}_x + fA_{x+t} - G\ddot{a}_{x+t}$
	Continuous	$S\bar{A}_{x+t} + e\bar{a}_{x+t} + f\bar{A}_x - G\bar{a}_{x+t}$
Endowment assurance	**Discrete**	$SA_{x+t:\overline{n-t}\vert} + e\ddot{a}_{x+t:\overline{n-t}\vert} + fA_{x+t:\overline{n-t}\vert} - G\ddot{a}_{x+t:\overline{n-t}\vert}$
	Continuous	$S\bar{A}_{x+t:\overline{n-t}\vert} + e\bar{a}_{x+t:\overline{n-t}\vert} + f\bar{A}_{x+t:\overline{n-t}\vert} - G\bar{a}_{x+t:\overline{n-t}\vert}$
Last survivor assurance	both x and y are alive: ${}_tV_{\overline{x:y}} = A_{\overline{x+t:y+t}} - P_{\overline{x:y}}\ddot{a}_{\overline{x+t:y+t}}$	
	y had previously died: ${}_tV_{\overline{x:y}} = A_{x+t} - P_{\overline{x:y}}\ddot{a}_{x+t}$	

Retrospective reserves

Gross premium reserve: $\frac{l_x}{l_{x+t}}(1+i)^t\left\{G\ddot{a}_{x:\overline{t}\vert}^{(m)} - S\bar{A}^1_{x:\overline{t}\vert} - I - e\ddot{a}_{x:\overline{t}\vert}^{(m)} - f\bar{A}^1_{x:\overline{t}\vert}\right\}$

If: 1. the retrospective and prospective reserves are calculated on the same basis; and
2. this basis is the same as the basis used to calculate the premiums used in the reserve calculation

then the retrospective reserve will be equal to the prospective reserve.

RECURSIVE FORMULA

Gross premium Reserve: $\left({}_tV' + G - e\right)(1+i) - q_{x+t}(S+f) = \left(1 - q_{x+t}\right){}_{t+1}V'$

Profit over the year: $PRO_t = \left({}_tV' + G - e\right)(1+i) - q_{x+t}(S+f) - \left(1 - q_{x+t}\right){}_{t+1}V'$

NET PREMIUM

Discrete: $P_x = \dfrac{A_x}{\ddot{a}_x}$ $\quad P^{\ \ 1}_{x:\overline{n}|} = \dfrac{A^{\ \ 1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$ $\quad P_{x:\overline{n}|} = \dfrac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$

Continuous: $\bar{P}\left(\bar{A}_x\right) = \dfrac{\bar{A}_x}{\bar{a}_x}$ $\quad \bar{P}\left(\bar{A}^{\ \ 1}_{x:\overline{n}|}\right) = \dfrac{\bar{A}^{\ \ 1}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$ $\quad \bar{P}\left(\bar{A}_{x:\overline{n}|}\right) = \dfrac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$

Net premium reserve

Whole life assurance $\quad {}_tV_x = 1 - \dfrac{\ddot{a}_{x+t}}{\ddot{a}_x} = \dfrac{A_{x+t} - A_x}{1 - A_x}$

Endowment assurance $\quad {}_tV_x = 1 - \dfrac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} = \dfrac{A_{x+t:\overline{n-t}|} - A_{x:\overline{n}|}}{1 - A_{x:\overline{n}|}}$

MORTALITY PROFIT

Death strain at risk (DSAR):

$$\text{DS} = \begin{cases} 0 & \text{if the life survives to } t+1 \\ (S - {}_{t+1}V) & \text{if the life dies in the year } [t, t+1) \end{cases}$$

$\text{Max}(S - {}_{t+1}V)$ is death strain at risk

Recursive relationship: $\left({}_tV + P\right)(1+i) = q_{x+t}S + p_{x+t\,t+1}V = {}_{t+1}V + q_{x+t}\left(S - {}_{t+1}V\right)$

Expected death strain (EDS): $\text{EDS} = q_{x+t}\left(S - {}_{t+1}V\right)$

Actual death strain (ADS):

$$\text{ADS} = \begin{cases} 0 & \text{if the life survives to } t+1 \\ (S - {}_{t+1}V) & \text{if the life dies in the year } [t, t+1) \end{cases}$$

Mortality profit: Monthly Profit = Expected Death Strain − Actual Death Strain

Mortality profit on a portfolio of policies

$$\begin{aligned}
\text{Total DSAR} &= \sum_{\text{all policies}} \left(S - {}_{t+1}V\right) \\
\text{Total EDS} &= \sum_{\text{all policies}} q_{x+t}\left(S - {}_{t+1}V\right) \\
&= q_{x+t}\left(\sum_{\text{all policies}} \left(S - {}_{t+1}V\right)\right) \\
&= q_{x+t}(\text{total DSAR}) \\
\text{Total ADS} &= \sum_{\text{death claims}} \left(S - {}_{t+1}V\right) \\
\text{Mortality Profit} &= \text{total EDS} - \text{total ADS}
\end{aligned}$$

DS payable immediately:

$$\text{DS} = \begin{cases} 0 & \text{if the life survives to } t+1 \\ \left(S(1+i)^{1/2} - {}_{t+1}V\right) & \text{if the life dies in the year } [t, t+1) \end{cases}$$

Allowing for survival benefits

Recursive relationship: $\left({}_tV + P\right)(1+i) = q_{x+t}S + p_{x+t}\left({}_{t+1}V + R\right) = {}_{t+1}V + R + q_{x+t}\left(S - \left({}_{t+1}V + R\right)\right)$

DS:

$$\text{DS} = \begin{cases} 0 & \text{if the life survives to } t+1 \\ \left(S - \left({}_{t+1}V + R\right)\right) & \text{if the life dies in the year } [t, t+1) \end{cases}$$

EDS: $\text{EDS} = q_{x+t}\left(S - \left({}_{t+1}V + R\right)\right)$

PROFIT TESTING

Evaluating expected cashflows

- Premiums received and their times of payment
- Expected expenses (from the basis) and their times of payment
- Contingent benefits payable under the contract
- Other benefits payable under the contract
- Other expected cash payments
- Other expected cash receipts
- The reserves required for a contract

Example: Whole life assurance

Income	Premiums	P (from data)
	Interest on Reserves	$i \cdot S \cdot {}_tV$
	Interest on Balances	$(P-e)i$
Expenditure	Expenses	e (from data)
	Expected Surrender Value	$(aq)^w_{[x]+t} \cdot (SV)_{t+1}$
	Expected Death Claims	$(aq)^d_{[x]+t} \cdot S$
	Transfer to Reserves	$(ap)_{[x]+t} \cdot S \cdot {}_{t+1}V - S \cdot_t V$
	Profit	Balancing item

Profit vector: $(PRO)_t$ — $(PRO)_0$ usually contains pre-contract expenses only.

Profit signature: $(PS)_t = {}_{t-1}(ap)_x (PRO)_t$ — Note that $\Pi_0 = (PRO)_0$.

Internal rate of return: IRR — Such that $\sum_{k=0}^{n} \frac{(PS)_t}{(1+IRR)^k} = 0$.

Net present value: $NPV = \sum_{t=1}^{\infty} (1+i_d)^{-t} (PS)_t$

Profit margin: $M = \dfrac{NPV}{EPV(\text{Premiums})}$

Zeroising Negative Cashflows: The process of calculation of the non-unit reserve

Single financing phase at outset: The profit signature has a single negative value (funds are provided by the insurance company) at policy duration zero

Cashflows:

1. Equation of Value: $(\text{NUCF})_t + {}_{t-1}V\,(1+i_s) - (ap)_{x+t-1}V = (\text{PRO})_t$
2. m : the greatest duration t for which $(\text{NUCF})_t$ is negative.
3. ${}_tV = 0$ for $t \geq m$.
4. ${}_{m-1}V = -\dfrac{(\text{NUCF})_m}{(1+i_s)}$
5. Formula for adjusted cashflow:

$$(\text{NUCF})'_{m-1} = (\text{NUCF})_{m-1} - (ap)_{x+m-2\;m-1}V$$

6. Choose one of the 2 paths depending on whether the adjusted cashflow is positive or negative:

a) If $(\text{NUCF})'_{m-1} > 0$, then:

$$(\text{PRO})_{m-1} = (\text{NUCF})'_{m-1}$$

b) If $(\text{NUCF})'_{m-1} < 0$, then we repeat the process establishing non-unit reserves ${}_{m-2}V$ at policy duration $m-2$, then

$$(\text{NUCF})'_{m-1} + {}_{m-2}V\,(1+i_s) = (\text{PRO})_{m-1}$$

and choose ${}_{m-2}V$ so that $(\text{PRO})_{m-1} = 0$, i.e.

$$_{m-2}V = -\frac{(\text{NUCF})'_{m-1}}{(1+i_s)}$$