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A4.7 Severity Models

SEVERITY DISTRIBUTIONS

Distribution	Probability density function	Formulas worth memorizing		
Uniform	$f(x)=\frac{1}{b-a},\ a\le x\le b$	$F(x)=\frac{x-a}{b-a}$	$E[X]=\frac{a+b}{2}$	$Var(X)=\frac{(b-a)^2}{12}$
Exponential	$f(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}},\ x>0$	$F(x)=1-e^{-\frac{x}{\theta}}$	$E[X]=\theta$	$Var(X)=\theta^2$
Weibull	$f(x)=\frac{\tau}{x}\left(\frac{x}{\theta}\right)^{\tau}e^{-\left(\frac{x}{\theta}\right)^{\tau}},\ x>0$	$F(x)=1-e^{-\left(\frac{x}{\theta}\right)^{\tau}}$		
Gamma	$f(x)=\frac{\left(\frac{1}{\theta}\right)^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\theta}},\ x>0$	$F(x)=\Pr(X^*\ge\alpha)$ X^* is Poisson with $\lambda=\frac{x}{\theta}$ If α is an integer.	$E[X]=\alpha\theta$	$Var(X)=\alpha\theta^2$
Beta	$f(x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1},$ $0<x<1$		$E[X]=\frac{a}{a+b}$ If a and b are integers.	$E\left[X^2\right]=\frac{a(a+1)}{(a+b)(a+b+1)}$ If a and b are integers.
Pareto	$f(x)=\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}},\ x>0$	$F(x)=1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}$	$E[X]=\frac{\theta}{\alpha-1}$ If $\alpha>1$ is an integer.	
Single P. Pareto	$f(x)=\frac{\alpha\theta^{\alpha}}{x^{\alpha+1}},\ x>\theta$	$F(x)=1-\left(\frac{\theta}{x}\right)^{\alpha}$	$E[X]=\frac{\alpha\theta}{\alpha-1}$	
Lognormal	$f(x)=\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\log x-\mu)^2}{2\sigma^2}},\ x>0$	$F(x)=N\left(\frac{\log x-\mu}{\sigma}\right)$	$E[X]=e^{\mu+\frac{\sigma^2}{2}}$	$E\left[X^2\right]=e^{2\mu+2\sigma^2}$

Distribution	Moment Generating Function
Exponential	$M_X(t)=\frac{\lambda}{\lambda-t},\quad t<\lambda$
Weibull	$M_X(t)=\int_0^{\infty}e^{tx}\cdot\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-(x/\beta)^{\alpha}}\,dx$
Gamma	$M_X(t)=\left(\frac{1}{1-\theta t}\right)^{\alpha},\quad t<\frac{1}{\theta}$

EXTREME VALUE DISTRIBUTIONS

Fréchet distribution $F(x)=\exp\left(-\left(1+\xi\left(\frac{x-\mu}{\theta}\right)\right)^{-1/\xi}\right),\quad \xi>0$ and $x>\mu-\frac{\theta}{\xi}$

Gumbel Distribution: $F(x)=\exp\left(-\exp\left(-\frac{x-\mu}{\theta}\right)\right)$

Weibull EV distribution: $F(x)=\exp\left(-\left(1+\xi\left(\frac{x-\mu}{\theta}\right)\right)^{-1/\xi}\right),\quad \xi<0$ and $x<\mu-\frac{\theta}{\xi}$

Generalized Pareto Distribution :
$$G_{\xi,\beta}(x)=\begin{cases}1-(1+\xi x/\beta)^{-1/\xi} & \xi\neq 0\\ 1-e^{-x/\beta} & \xi=0\end{cases}\quad\text{where}\quad \begin{array}{c}\beta>0,\ x\ge 0 \text{ for } \xi\ge 0,\\ \text{and } 0\le x\le -\beta/\xi \text{ for } \xi<0\end{array}$$

SCALING & TRANSFORMATION

Scaling:	$Y = cX$	$\rightarrow$	$F_Y(y) = F_X\left(\frac{y}{c}\right)$	
Transformation:	$Y = g(X)$	$\rightarrow$	$F_Y(y) = F_X\left(g^{-1}(y)\right)$	If g is monotonically increasing.
		$\rightarrow$	$F_Y(y) = S_X\left(g^{-1}(y)\right)$	If g is monotonically decreasing.
Constant Multiple:	$Y = cX$	$\rightarrow$	$f_Y(y) = f_X(k(y)) \times \lvert k'(y)\rvert = f_X\left(\frac{y}{c}\right) \times \frac{1}{c}$.	
Power:	$Y = X^{1/\tau}$	$\rightarrow$	$f_Y(y) = \tau y^{\tau-1} f_X\left(y^{\tau}\right)$	
Exponential:	$Y = g(X)$	$\rightarrow$	$f_Y(y) = \frac{1}{y} \cdot f_X(\log y)$	

MIXING & SPLICING

Mixture: $f_X(x) = w_1 f_2(x) + w_2 f_2(x) + \cdots + w_k f_k(x)$ Where $w_1 + w_2 + \cdots + w_k = 1$.

Splices:

$$f_X(x) = \begin{cases} w_1 f_1(x), & x_0 \le x < x_1 \\ w_2 f_2(x), & x_1 \le x < x_2 \\ \quad \cdots & \\ w_k f_k(x), & x_{k-1} \le x < x_k \end{cases}$$

TAILS & LIMITING DISTRIBUTIONS

Limit of ratio of two survival functions: $\lim_{x\to\infty} \frac{S_1(x)}{S_2(x)} = \lim_{x\to\infty} \frac{f_1(x)}{f_2(x)}$

Limit of hazard rate function: $\lim_{x\to\infty} h(x) = \lim_{x\to\infty} \frac{f(x)}{S(x)}$

Limit of mean excess loss function: $\lim_{d\to\infty} e_X(d) = \lim_{d\to\infty} \frac{\int_d^\infty S(x)dx}{S(d)} = \lim_{d\to\infty} \frac{1}{h(d)}$

Equilibrium distribution: $f_e(x) = \frac{S(x)}{E[X]} \quad \rightarrow \quad E\left[X_e\right] = \frac{E\left[X^2\right]}{2E[X]}$

A4.8 Frequency Models

FREQUENCY DISTRIBUTIONS

Distribution	Probability mass function	Formulas worth memorizing	
Poisson	$P(n) = \frac{e^{-\lambda}\lambda^n}{n!}, n = 0, 1, 2, \ldots, \infty$	$E[N] = \lambda$	$Var(N) = \lambda$
Binomial	$P(n) = \binom{m}{n} q^n (1-q)^{m-n}, n = 0, 1, 2, \ldots, m$	$E[N] = mq$	$Var(N) = mq(1-q)$
Geometric	$P(n) = \left(\frac{1}{1+\beta}\right)\left(\frac{\beta}{1+\beta}\right)^n, n = 0, 1, 2, \ldots, \infty$	$E[N] = \beta$	$Var(N) = \beta(1+\beta)$
Negative Binomial	$P(n) = \binom{n+r-1}{n}\left(\frac{1}{1+\beta}\right)^r\left(\frac{\beta}{1+\beta}\right)^n, n = 0, 1, 2, \ldots, \infty$	$E[N] = r\beta$	$Var(N) = r\beta(1+\beta)$

Distribution	Probability Generating Function	Moment Generating Function
Poisson	$P_N(t) = e^{\lambda(t-1)}$	$M_N(t) = e^{\lambda(e^t-1)} = P_N\left(e^t\right)$
Binomial	$P_N(t) = (1 + q(t-1))^m$	$M_N(t) = \left(1 + q\left(e^t - 1\right)\right)^m$
Geometric	$P_X(t) = \frac{p}{1-(1-p)t}$	$P_N(t) = p\sum_{k=0}^{\infty}\left((1-p)e^t\right)^k$
Negative Binomial	$P_N(t) = \frac{1}{[1-\beta(t-1)]^r}$	$M_N(t) = \frac{1}{[1-\beta(e^t-1)]^r}$

GENERAL CLASS OF DISTRIBUTIONS

The (a,b,0) class: $\Pr\left(N^L = n\right) = p_n$ Poisson, Binomial, Negative Binomial, and Geometric distributions.

The (a,b,1) class: $p_n^T = \frac{p_n}{1-p_0}$ **Zero-truncated distributions**.

$p_n^M = \left(1 - p_0^M\right)\left(\frac{p_n}{1-p_0}\right)$ **Zero-modified distributions**.

A4.9 Aggregate Models

AGGREGATE MODELS

Individual risk model: $S = X_1 + X_2 + \cdots + X_n \rightarrow E[S] = nE[X]$

$\rightarrow Var(S) = nVar(X)$

Collective risk model: $S = X_1 + X_2 + \cdots + X_N \rightarrow E[S] = E[N]E[X]$

$\rightarrow Var(S) = E[N]Var(X) + Var(N)E[X]^2$

Normal Distribution: $X_i \overset{iid}{\sim} N\left(\mu,\ \sigma^2\right) \rightarrow S^L = X_1 + X_2 + \cdots + X_n \sim N\left(n\mu,\ n\sigma^2\right)$

Exponential Distribution: $X_i \overset{iid}{\sim} Exp\left(\theta\right) \rightarrow S^L = X_1 + X_2 + \cdots + X_n \sim Gamma\left(\alpha = n,\ \theta\right)$

Normal approximation: $S \dot{\sim} N\left(\mu = E[S], \sigma^2 = Var(S)\right) \rightarrow Pr(S \le k) \approx N\left(\frac{k-\mu}{\sigma}\right)$

A4.10 Coverage Modifications

PAYMENT PER LOSS

Policy	Payment per loss	Expected payment per loss
With ordinary deductible d	$Y^L = \begin{cases} 0, & X < d \\ X - d, & X \ge d \end{cases}$	$E\left[Y^L\right] = E[X] - E[X \wedge d]$
With franchise deductible d^*	$Y^L = \begin{cases} 0, & X \le d^* \\ X, & X > d^* \end{cases}$	$E\left[Y^L\right] = E\left[X \mid X > d^*\right]$
With maximum covered loss u	$Y^L = \begin{cases} X, & X \le u \\ u, & X > u \end{cases}$	$E\left[Y^L\right] = E[X \wedge u]$
With d and u	$Y^L = \begin{cases} 0, & X \le d \\ X - d, & d < X \le u \\ u - d, & X > u \end{cases}$	$E\left[Y^L\right] = E[X \wedge u] - E[X \wedge d]$
With $d,\ u$ and coinsurance factor α	$Y^L = \begin{cases} 0, & X \le d \\ \alpha(X - d), & d < X \le u \\ \alpha(u - d), & X > u \end{cases}$	$E\left[Y^L\right] = \alpha(E[X \wedge u] - E[X \wedge d])$
With d, u, α and inflation rate r	$Y^L = \begin{cases} 0, & X \le \frac{d}{1+r} \\ \alpha(1+r)\left(X - \frac{d}{1+r}\right), & \frac{d}{1+r} < X \le \frac{u}{1+r} \\ \alpha(1+r)\left(\frac{u}{1+r} - \frac{d}{1+r}\right), & X > \frac{u}{1+r} \end{cases}$	$E\left[Y^L\right] = \alpha(1+r)\left(E\left[X \wedge \frac{u}{1+r}\right] - E\left[X \wedge \frac{d}{1+r}\right]\right)$

Loss elimination ratio: $\text{LER} = 1 - \frac{E\left[Y^L\right]}{E[X]}$

PAYMENT PER PAYMENT

Payment per Payment $Y^P = Y^L | X > d \quad \rightarrow \quad E\left[Y^P\right] = \frac{E\left[Y^L\right]}{\Pr(X>d)}$

Policy	Loss	Payment per payment
With ordinary deductible d	$X \sim Unif(0,b)$	$Y^P \sim Unif(0, b-d)$
	$X \sim Exp(\theta)$	$Y^P \sim Exp(\theta)$
	$X \sim Pareto(\alpha, \theta)$	$Y^P \sim Pareto(\alpha, \theta + d)$

REINSURANCE

Proportional reinsurance	Insurer pays	Reinsurer pays
With quota share α	$Y = (1-\alpha)X$	$Y = \alpha X$
With retention u and surplus share α	$Y = \begin{cases} X, & X \le u \\ u + (1-\alpha)(X-u), & X > u \end{cases}$	$Y = \begin{cases} 0, & X \le u \\ \alpha(X-u), & X > u \end{cases}$

Excess of loss reinsurance	Insurer pays	Reinsurer pays
Covers losses above d	$Y = \begin{cases} X, & X \le d \\ d, & X > d \end{cases}$	$Y = \begin{cases} 0, & X \le d \\ X-d, & X > d \end{cases}$
Covers losses above d but below u	$Y = \begin{cases} X, & X \le d \\ d, & d \le X < u \\ X+d-u, & X > u \end{cases}$	$Y = \begin{cases} 0, & X \le d \\ X-d, & d \le X < u \\ u-d, & X > u \end{cases}$

A4.11 Risk Measures

RISK MEASURES

Coherent risk measures:

1. **Translation Invariance:** $\rho(X+c) = \rho(X) + c$
2. **Positive Homogeneity:** $\rho(cX) = c\rho(X)$
3. **Subadditivity:** $\rho(X+Y) \le \rho(X) + \rho(Y)$
4. **Monotonicity:** $\rho(X) \le \rho(Y)$ If $\Pr(X \le Y) = 1$

Value-at-Risk: $\mathrm{VaR}_\alpha(X)$ Where $\Pr(X \le \mathrm{VaR}_\alpha(X)) = \alpha$.

Tail-Value-at-Risk: $\mathrm{TVAR}_\alpha(X) = E[X | X > \mathrm{VaR}_\alpha(X)] = \mathrm{VaR}_\alpha(X) + \frac{E[X] - E[X \wedge \mathrm{VaR}_\alpha(X)]}{1-\alpha}$

A4.12 Construction and Selection of Parametric Models

MLE WITH COMPLETE DATA

For distributions that belong to the exponential family:

1. Determine $L(\theta)$.
2. Apply natural logarithm, obtain $l(\theta) = \log L(\theta)$.
3. Take the first derivative with respect to the parameter, obtain $l'(\theta)$.
4. Set $l'(\theta) = 0$, obtain $\hat{\theta}$, which is the MLE.

Distribution	Likelihood Function	Maximum likelihood estimate(s)	
Exponential	$L(\theta) = f(x_1)\dots f(x_n)$	$\hat{\theta} = \bar{x}$	
Normal		$\hat{\mu} = \bar{x}$	$\hat{\sigma}^2 = \frac{1}{n}\sum(x_i - \hat{\mu})^2$
Lognormal		$\hat{\mu} = \frac{1}{n}\sum \log x_i$	$\hat{\sigma}^2 = \frac{1}{n}\sum(\log x_i - \hat{\mu})^2$
Uniform		$\hat{b} = \max(x_1, \dots, x_n)$	
Binomial	$L(\theta) = p(x_1)\dots p(x_n)$	$\hat{q} = \frac{\bar{x}}{m}$	
Poisson		$\hat{\lambda} = \bar{x}$	

MLE WITH INCOMPLETE DATA

	Likelihood function	Note
Grouped data	$L = (F(c_1) - F(c_0))^{m_1} \dots (F(c_n) - F(c_{n-1}))^{m_n}$	Where $c_0 < c_1 < \dots < c_n$ are interval boundaries.
Left-truncated data	$L = \frac{f(x_1)}{S(d)} \dots \frac{f(x_n)}{S(d)}$	Losses below d are not reported.
Right-censored data	$L = f(x_1)\dots f(x_n)\, S(u)^m$	Losses are capped at u.
Left-truncated & Right-censored data	$L = \frac{f(x_1)}{S(d)} \dots \frac{f(x_n)}{S(d)} \left(\frac{S(u)}{S(d)}\right)^m$	Losses below d are not reported. Losses are capped at u.

VARIANCE OF MLE

Number of parameters	Information matrix	Variance of MLE
1	$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} l(\theta)\right]$	$\text{Var}(\hat{\theta}) = I(\theta)^{-1}$
2	$I(\theta_1, \theta_2) = -E\begin{bmatrix} \frac{\partial^2}{\partial\theta_1^2} l(\theta_1,\theta_2) & \frac{\partial^2}{\partial\theta_1\partial\theta_2} l(\theta_1,\theta_2) \\ \frac{\partial^2}{\partial\theta_1\partial\theta_2} l(\theta_1,\theta_2) & \frac{\partial^2}{\partial\theta_2^2} l(\theta_1,\theta_2) \end{bmatrix}$	$VCOV\left(\hat{\theta}_1, \hat{\theta}_2\right) = I(\theta_1, \theta_2)^{-1} = \begin{bmatrix} \text{Var}\left(\hat{\theta}_1\right) & \text{Cov}\left(\hat{\theta}_1, \hat{\theta}_2\right) \\ \text{Cov}\left(\hat{\theta}_1, \hat{\theta}_2\right) & \text{Var}\left(\hat{\theta}_2\right) \end{bmatrix}$

Inverting a matrix: $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

Normal CI for θ: $\hat{\theta} \pm z\sqrt{\widehat{\text{Var}}(\hat{\theta})}$ Where $\widehat{\text{Var}}(\hat{\theta})$ is an estimate of $\text{Var}(\hat{\theta})$.

DELTA METHOD

Number of parameters	Function of MLE	Variance using Delta method
1	g is a function of θ $\hat{g}$ is an estimator of g, using $\hat{\theta}$.	$\mathrm{Var}(\hat{g}) \approx \mathrm{Var}(\hat{\theta}) \left(\frac{\partial g}{\partial \theta}\right)^2$
2	h is a function of θ_1 and θ_2. $\hat{h}$ is an estimator of h, using $\hat{\theta}_1$ and $\hat{\theta}_2$.	$\mathrm{Var}(\hat{h}) \approx \mathrm{Var}\left(\hat{\theta}_1\right)\left(\frac{\partial h}{\partial \theta_1}\right)^2 + \mathrm{Var}\left(\hat{\theta}_2\right)\left(\frac{\partial h}{\partial \theta_2}\right)^2$ $+2\,\mathrm{Cov}\left(\hat{\theta}_1, \hat{\theta}_2\right)\left(\frac{\partial h}{\partial \theta_1}\right)\left(\frac{\partial h}{\partial \theta_2}\right)$

CHI-SQUARE TEST

Null hypothesis: The parametric model fits its data well / The mean is the same across categories

Suppose there are k categories:

O_i is the observed value for category i.

Test statistic: $Q = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2(k-1)$

E_i is the expected value for category i.

1-tailed test. Reject H_0 if $Q > c$.

Suppose there are $k_1 \times k_2$ categories:

Test statistic: $Q = \sum_{j=1}^{k_2} \sum_{i=1}^{k_1} \frac{(o_{ij} - E_{ij})^2}{E_{ij}} \sim \chi^2\left((k_1 - 1)(k_2 - 1)\right)$

1-tailed test. Reject H_0 if $Q > c$.

Note: Subtract additional 1 degree of freedom for each parameter fitted from the data.

LIKELIHOOD RATIO TEST

$\boldsymbol{H_0 : \theta = \theta_0}$ **vs** $\boldsymbol{H_1 : \theta \neq \theta_0}$

$R = \frac{L(X|\theta_0)}{L(X|\theta_1)}$

Where θ_1 can be the maximum likelihood estimate of θ.

Test Statistic $-2\log R \sim \chi^2(1)$

The DOF depends on the number of parameters specified in H_0 and H_1.

1-tailed test. Reject H_0 if test statistic $> c$.

SBC & AIC

Schwartz Bayesian Criterion (also known as BIC): $\log L - \frac{r}{2}\log n$

r is the number of parameters in the model.

Akaike Information Criterion (AIC): $\log L - r$

Note: The model with higher SBC(BIC)/AIC is better.

GRAPHICAL METHODS

D(x) plots: x on the x axis.

$D(x) = F_n(x) - F^*(x)$ on the y axis.

$F_n(x)$ is the empirical distribution function.

$F^*(x)$ is the fitted distribution function.

p-p plots: $F_n(x)$ on the x axis. $F^*(x)$ on the y axis.

A4.13 Credibility

LIMITED FLUCTUATION CREDIBILITY

We want...to be within k of the mean p of the time.	**Range parameter, k** Note: $z = N^{-1}\left(\frac{1+p}{2}\right)$	**Total number of exposures needed, e_F**	**Total number of claims needed, n_F**
Average number of claims	$k = \frac{z\sqrt{\frac{\text{Var}(N)}{e_F}}}{E[N]}$	$e_F = \left(\frac{z}{k}\right)^2 \times \frac{\text{Var}(N)}{E[N]^2}$	$n_F = \left(\frac{z}{k}\right)^2 \times \frac{\text{Var}(N)}{E[N]^2} \times E[N]$
Total number of claims	$k = \frac{z\sqrt{e_F \text{Var}(N)}}{e_F E[N]}$		
Average claim size	$k = \frac{z\sqrt{\frac{\text{Var}(X)}{n_F}}}{E[X]}$	$e_F = \left(\frac{z}{k}\right)^2 \times \frac{\text{Var}(X)}{E[X]^2} \times \frac{1}{E[N]}$	$n_F = \left(\frac{z}{k}\right)^2 \times \frac{\text{Var}(X)}{E[X]^2}$
Average aggregate claims	$k = \frac{z\sqrt{\frac{\text{Var}(S)}{e_F}}}{E[S]}$	$e_F = \left(\frac{z}{k}\right)^2 \times \frac{\text{Var}(S)}{E[S]^2}$	$n_F = \left(\frac{z}{k}\right)^2 \times \frac{\text{Var}(S)}{E[S]^2} \times E[N]$
Total aggregate claims	$k = \frac{z\sqrt{e_F \text{Var}(S)}}{e_F E[S]}$		

COMPOUND POISSON APPLICATION

We want...to be within k of the mean p of the time.	**Frequency**	**Total number of exposures needed, e_F**	**Total number of claims needed, n_F**
Average number of claims	$N \sim Poisson(\lambda)$	$e_F = \left(\frac{z}{k}\right)^2 \times \frac{1}{\lambda}$	$n_F = \left(\frac{z}{k}\right)^2$
Total number of claims			
Average claim size		$e_F = \left(\frac{z}{k}\right)^2 \times \frac{\text{Var}(X)}{E[X]^2} \times \frac{1}{\lambda}$	$n_F = \left(\frac{z}{k}\right)^2 \times \frac{\text{Var}(X)}{E[X]^2}$
Average aggregate claims		$e_F = \left(\frac{z}{k}\right)^2 \times \left(1 + \frac{\text{Var}(X)}{E[X]^2}\right) \times \frac{1}{\lambda}$	$n_F = \left(\frac{z}{k}\right)^2 \times \left(1 + \frac{\text{Var}(X)}{E[X]^2}\right)$
Total aggregate claims			

PARTIAL CREDIBILITY

Credibility factor: $Z = \sqrt{\frac{\text{Number of exposures available}}{\text{Number of exposures needed for full credibility}}}$

or $Z = \sqrt{\frac{\text{Number of claims available}}{\text{Number of claims needed for full credibility}}}$

Credibility premium: $P = Z \times \text{Observation } + (1 - Z) \times \text{Manual Rate}$

BAYESIAN CREDIBILITY

Observations: $\vec{X} = \{X_1 = x_1, \ldots, X_n = x_n\}$

Prior pmf: $\Pr(\theta = k)$ if θ is discrete.

Prior pdf: $f(\theta = k)$ if θ is continuous.

Posterior pmf/pdf

	$\boldsymbol{X}$ is discrete	$\boldsymbol{X}$ is continuous
$\boldsymbol{\theta}$ **is discrete**	$\Pr(\theta=k \mid \vec{\boldsymbol{X}}) = \frac{\Pr(\theta=k)\Pr(\vec{\boldsymbol{X}}\mid\theta=k)}{\Pr(\vec{\boldsymbol{X}})}$	$\Pr(\theta=k \mid \vec{\boldsymbol{X}}) = \frac{\Pr(\theta=k)f(\vec{\boldsymbol{X}}\mid\theta=k)}{f(\vec{\boldsymbol{X}})}$
$\boldsymbol{\theta}$ **is continuous**	$f(\theta=k \mid \vec{\boldsymbol{X}}) = \frac{f(\theta=k)\Pr(\vec{\boldsymbol{X}}\mid\theta=k)}{\Pr(\vec{\boldsymbol{X}})}$	$f(\theta=k \mid \vec{\boldsymbol{X}}) = \frac{f(\theta=k)f(\vec{\boldsymbol{X}}\mid\theta=k)}{f(\vec{\boldsymbol{X}})}$

Predictive Probability $\Pr\left(X_{n+1} \le x \mid \vec{\boldsymbol{X}}\right) = \sum \Pr(X \le x \mid \theta = k)\Pr(\theta = k \mid \vec{\boldsymbol{X}})$ If θ is discrete.

$\Pr\left(X_{n+1} \le x \mid \vec{\boldsymbol{X}}\right) = \int \Pr(X \le x \mid \theta = k) f(\theta = k \mid \vec{\boldsymbol{X}}) dk$ If θ is continuous.

Bayesian Premium $E\left[X_{n+1} \mid \vec{\boldsymbol{X}}\right] = \sum E[X \mid \theta = k]\Pr(\theta = k \mid \vec{\boldsymbol{X}})$ If θ is discrete.

$E\left[X_{n+1} \mid \vec{\boldsymbol{X}}\right] = \int E[X \mid \theta = k] f(\theta = k \mid \vec{\boldsymbol{X}}) dk$ If θ is continuous.

Distribution	Conjugate prior	Posterior
$X \mid \lambda \sim \text{Poisson}(\lambda)$	$\lambda \sim \text{Gamma}(\alpha, \theta)$	$\lambda \mid \vec{\boldsymbol{X}} \sim \text{Gamma}\left(\alpha^* = \alpha + \sum X_i, \theta^* = \frac{1}{\frac{1}{\theta}+n}\right)$
$X \mid \mu \sim \text{Normal}\left(\mu, \sigma^2\right)$	$\mu \sim \text{Normal}(m, v)$	$\mu \mid \vec{\boldsymbol{X}} \sim \text{Normal}\left(m^* = \frac{\sigma^2 m + nv\bar{X}}{\sigma^2+nv}, v^* = \frac{\sigma^2 v}{\sigma^2+nv}\right)$
$X \mid q \sim \text{Bernoulli}\ (q)$	$q \sim \text{Beta}(a, b)$	$q \mid \vec{\boldsymbol{X}} \sim \text{Beta}\left(a^* = a + \sum X_i, b^* = b + n - \sum X_i\right)$
$X \mid Y \sim \text{Exponential}\ \left(\theta = \frac{1}{Y}\right)$	$Y \sim \text{Gamma}(\alpha, \theta)$	$Y \mid \vec{\boldsymbol{X}} \sim \text{Gamma}\left(\alpha^* = \alpha + n, \theta^* = \frac{1}{\frac{1}{\theta}+\sum x_i}\right)$

BÜHLMANN CREDIBILITY

Expected Value of Hypothetical Mean: $\mu = E\left[E\left[X|\theta\right]\right]$

Expected Value of Process Variance: $v = E\left[Var\left(X|\theta\right)\right]$

Variance of Hypothetical Mean: $a = Var\left(E\left[X|\theta\right]\right)$

Credibility Factor: $Z = \dfrac{n}{n + \dfrac{v}{a}}$

Bühlmann Premium: $E\left[X_{n+1} \middle| \vec{X}\right] = Z\bar{X} + (1 - Z)\,\mu$

Note: The Bühlmann estimate is a linear approximation of the Bayesian estimate. If the Bayesian estimate is a linear function of the sample mean, then the Bühlmann estimate is equal to the Bayesian estimate.

NON-PARAMETRIC EMPIRICAL BAYES CREDIBILITY

Equal Sample Size: $\bar{X}_i = \dfrac{\sum_{j=1}^{n} X_{ij}}{n}$

$$\hat{\mu} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n} X_{ij}}{rn} \qquad \hat{v} = \frac{\sum_{i=1}^{r}\sum_{j=1}^{n}\left(X_{ij} - \bar{X}_i\right)^2}{r\,(n-1)} \qquad \hat{a} = \frac{\sum_{i=1}^{r}\left(\bar{X}_i - \hat{\mu}\right)^2}{r-1} - \frac{\hat{v}}{n}$$

Credibility factor: $Z = \dfrac{n}{n + \dfrac{\hat{v}}{\hat{a}}}$

Credibility premium: $Z\bar{X}_i + (1 - Z)\,\hat{\mu}$

Unequal Sample Size: $\bar{X}_i = \dfrac{\sum_{j=1}^{n_i} m_{ij} X_{ij}}{m_i}$

$\hat{\mu} = \dfrac{\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij} X_{ij}}{m}$ $\hat{v} = \dfrac{\sum_{i=1}^{r}\sum_{j=1}^{n_i} m_{ij}\left(X_{ij} - \bar{X}_i\right)^2}{\sum_{i=1}^{r}(n_i - 1)}$ $\hat{a} = \dfrac{\sum_{i=1}^{r} m_i\left(\bar{X}_i - \hat{\mu}\right)^2 - \hat{v}\,(r-1)}{m - \frac{1}{m}\sum_{i=1}^{r} m_i^2}$

Credibility factor: $Z = \dfrac{m_i}{m_i + \frac{\hat{v}}{\hat{a}}}$

Credibility premium: $Z\bar{X}_i + (1 - Z)\,\hat{\mu}$

SEMI-PARAMETRIC EMPIRICAL BAYES CREDIBILITY

Poisson Model: $\hat{\mu} = \bar{X}$ $\hat{v} = \bar{X}$ $\hat{a} = S^2 - \hat{v}$

Credibility factor: $Z = \dfrac{n}{n + \frac{\hat{v}}{\hat{a}}}$

Credibility premium: $Z\bar{X}_i + (1 - Z)\,\hat{\mu}$

A4.14 Short-Term Insurance and Reinsurance Coverages

PROPERTY AND CASUALTY COVERAGES

Automobile Insurance:

- **Liability** and **medical benefit** coverage are often mandatory with regulated minimum coverage.
- Other coverages are **collision**, **uninsured motorist**, **other than collision**.
- Usually family and invited drivers are covered.
- Commercial auto insurance has a different price structure.
- Some states have at fault insurance in which the injured party must prove that the insured was at fault.
- Under **no fault insurance** the injured party is compensated by their own insurer.
- No fault increases personal injury premium and decreases liability premium.

Liability: Covers damage to another vehicle, and injury to another person, with various minimum coverage limitations to coverage.

Medical Benefit: Also provides income replacement to policyholder.

Uninsured and Underinsured Motorist: Provides protection to policyholder if injured by an uninsured or underinsured motorist.

Collision and Other-than-Collision Coverage:

- Optional covering damage to insured's vehicle, usually with a deductible (possibly recoverable).
- Insurers can sue in at fault jurisdictions (**subrogation**).
- Insurer can claim **salvage** value of vehicle when insured is paid full value.
- Premiums can vary by geographical area, age, gender and marital status of insured.
- **Comprehensive coverage** covers all perils except those specifically excluded.
- **Specified perils** cover only those listed.
- All risks coverage combines collisions and OTC.

Homeowners Insurance:

- **Doctrine of proximate cause**; peril must initiate unbroken sequence of events leading to a covered consequence.
- Deductible may be in effect for small loss.
- **Disappearing deductible**: deductible may disappear linearly for a larger loss. For example, for a deductible of D for losses below A with linearly disappearing deductible for losses between A and B ($A < B$), for a loss L between A and B the deductible is
$$\frac{B-L}{B-A} \times D,$$
and for a loss above B the deductible is 0.

Coinsurance:

- A policy may have a **coinsurance clause** that is activated if a loss and the insured value is less than some specified percentage of the full value of the property at the time of the loss; in case of a loss, the insurer will not pay more than the insured value, and will pay a fraction of the loss based on the fraction that insured value is of the coinsured value.
- Coinsurance adjusts for loss distribution skewed to smaller claims and reduces anti-selection (anti-selection occurs when underwriting deficiencies allow a high risk group to purchase insurance at the same price as a lower-risk group).

REINSURANCE

Reinsurance:

Reinsurance is insurance for an insurer. The primary insurer cedes part of its covered risk and a secondary insurer (or insurers) assume part of that risk. Reinsurers may reinsure further. Reinsurance is used for capital relief, increasing underwriting capacity, catastrophe protection, stabilizing loss experience, reducing risk concentration, taking advantage of a reinsurer's technical expertise for certain types of risk, and withdrawal from some markets.

Treaty Insurance: Under a treaty insurance contract the reinsurer covers the specified share of all the insurance policies issued by the ceding company. The share covered by the reinsurer may be different in different loss intervals. A loss interval under this arrangement may be called a **treaty layer**.

Facultative Insurance: Under a facultative insurance contract reinsurance is negotiated separately for each insurance policy that is reinsured.

Types of Reinsurance: The reinsurance may be

- **Proportional**, in which the reinsurer pays
  1. a specified proportion (**quota share**) of the loss, or
  2. a specified proportion above the amount retained by the primary insurer (**surplus share**).
- **Excess of Loss**, in which the reinsurer pays losses above the primary insurer's **retention** (a deductible, also called an **attachment point**, from the point of view of the reinsurer). Excess of loss reinsurance may be **per risk**, **per occurrence**, or **aggregate (also called stop loss)**.

Experience Rating: A reinsurer can calculate a loss ratio based on experience rating. The reinsurance payment would be found after applying trending, development and including **ALAE** (allocated adjustment expenses) to the actual losses and then applying the reinsurance terms (deductible, limit, and/or proportion). The **loss ratio** is the fraction of the reinsurance payment of the premium. The reinsurance premium would then include a provision for expenses and profit.

A4.15 Pricing and Reserving for Short-Term Insurance Coverages

RATEMAKING DATA

Exposure	Written exposure Earned exposure Unearned exposure
Premium	Written premium Earned premium Unearned premium
Claim	Reported claims Unreported claims
Loss	Reported loss = Paid loss + Case reserve IBNR reserve = Incurred but not reported reserve IBNER reserve = Incurred but not enough reported reserve Ultimate loss = Reported loss + IBNR reserve + IBNER reserve

Basic formulas:

$$\text{Frequency} = \frac{\text{Number of Claims}}{\text{Number of Exposures}}$$

$$\text{Severity} = \frac{\text{Losses}}{\text{Number of Claims}}$$

$$\textbf{Pure Premium/Loss Cost} = \frac{\text{Losses}}{\text{Number of Exposures}} = \text{Frequency} \times \text{Severity}$$

$$\textbf{Loss Ratio} = \frac{\text{Losses}}{\text{Premium}}$$

$$\textbf{Expense Ratio} = \frac{\text{Expenses}}{\text{Premium}}$$

LOSS RESERVING METHODS

Expected Loss Ratio Method	1. Ultimate Losses = Earned Premium × Expected Loss Ratio 2. Loss Reserve = Ultimate Losses − Paid Losses
Chain-ladder Method	1. Prepare a run-off triangle for paid losses. 2. Calculate age-to-age factors using average factor method or mean factor method. 3. Calculate age-to-ultimate factor f_{ULT}, which is the product of age-to-age factors. 4. Ultimate Losses = Paid Losses × f_{ULT} 5. Loss reserve = Ultimate Losses − Paid Losses
Bornhuetter -Ferguson method	1. Prepare a run-off triangle for paid losses. 2. Calculate age-to-age factors using average factor method or mean factor method. 3. Calculate age-to-ultimate factor f_{ULT}, which is the product of age-to-age factors. 4. Ultimate Losses = Paid Losses + Earned Premium × Expected Loss Ratio × $\left(1 - \frac{1}{f_{ULT}}\right)$ 5. Loss reserve = Ultimate Losses − Paid Losses
Discounted Loss Reserves	1. Project incremental payments using one of the above methods. 2. Discounted loss reserve = PV of projected incremental payments

AGGREGATION METHODS

Aggregation methods	Premium/Exposure	Loss
Calendar year	Transaction date	Transaction date
Calendar-accident year	Transaction date	Accident date
Policy year	Effective date	Effective date

PRICING FORMULA

Expense Ratio: All expenses other than loss adjustment expenses as a percentage of gross rate

Permissible loss ratio: $1 - \text{Expense Ratio}$

Gross Rate: $\dfrac{\text{Incurred Loss Cost per Unit of Exposure}}{\text{Permissible Loss Ratio}}$

Adjustments to data: $\text{Premium at current rates} = \text{Earned premium} \times \dfrac{\text{Current rate level}}{\text{Historical average rate level}}$

$\text{Ultimate losses} = \text{Reported losses} \times \text{Development factor}$

OVERALL RATE INDICATION

Loss cost method or Pure premium method	$\text{New Average Loss Cost} = \dfrac{\text{Expected Dollar Losses in Effective Period}}{\text{Number of Earned Exposure Units}}$ $\text{New Average Gross Rate} = \dfrac{\text{New Average Loss Cost} + \text{Fixed Expense per Exposure}}{\text{Permissible Loss Ratio}}$ $\text{Permissible Loss Ratio} = 1 - \text{Variable Expense Ratio} - \text{Profit and Contingencies Ratio}$
Loss ratio method	$\text{Expected Effective Loss Ratio} = = \dfrac{\text{Expected Dollar Losses in Effective Period}}{\text{Dollars of Earned Premium at Current Rates}}$ $\text{Fixed Expense Ratio} = \dfrac{\text{Fixed Expense Per Exposure}}{\frac{\text{Dollars of Earned Premium at Current Rates}}{\text{Number of Earned Exposure Units}}}$ $\text{Indicated Rate Change Factor} = \dfrac{\text{Expected Effective Loss Ratio} + \text{Fixed Expense Ratio}}{\text{Permissible Loss Ratio}}$ $\text{Indicated (or New) Average Gross Rate} = \dfrac{\text{Earned Premium at Current Rates}}{\text{Earned Exposure Units}}$ $\times(\text{Indicated Rate Change Factor})$

RISK CLASSIFICATION

Loss cost method: $\text{Indicated differential}_i = \dfrac{\text{Loss cost}_i}{\text{Loss cost}_{\text{base}}}$

Loss ratio method: $\text{Indicated differential}_i = \text{Existing differential}_i \times \dfrac{\text{Loss ratio}_i}{\text{Loss ratio}_{\text{base}}}$

Balancing back: $\text{Indicated overall rate} = \text{Current base rate} \times (1 + \text{Indicated overall rate change})$

$\text{Balance Back Factor} = \dfrac{\text{Average existing differential}}{\text{Average indicated differential}}$

$\text{Indicated rate}_i = \text{Indicated overall rate} \times \text{Balance Back Factor} \times \text{Indicated differential}_i$