# A4.1 Long-Term Insurance Coverages

## INSURANCE COVERAGES

**Term insurance**: Provides financial protection in case of the policyholder's death, for families or businesses

- **Convertibility**: Can be converted to whole life or endowment without new underwriting
- **Renewable term**: allows contract renewal without underwriting, but with higher premiums

**Whole life insurance**: Older lives can use it to cover funeral expenses or to reduce inheritance taxes, younger lives can use it as a passive investment vehicle with substantial death benefit

- ***Cash value***: If a policy lapses due to missed premium payments, the policyholder may receive a cash or surrender value

**Participating insurance**: Profits from invested premiums may be shared with the policyholder to enhance the contract's appeal, but any losses are not shared.

**Endowment insurance**: A hybrid of term insurance and fixed-term investment, often with cash value, available as participating or non-participating

**Riders**:

**Policy loans**: For policies with cash values, policyholders may borrow money from the insurer, using the cash value as collateral. Any outstanding loan will reduce the payout upon claim or policy surrender.

**Accelerated benefits**: The death benefit will be paid earlier than death if the policyholder is diagnosed with terminal illnesses.

**Accidental death benefit**: If the insured dies from accidents rather than through natural causes, the death benefit is increased.

**Waiver of premium**: Premiums may be waived during disability or illness (with medical proof) without causing the policy to lapse

**Family income benefit**: A rider on term insurance that pays an annuity-style benefit from the policyholder's death until the end of the original term, in addition to the death benefit.

**Joint life insurance**: Benefits are paid out upon the first death among insured individuals; it can be structured as term, endowment, or whole life, and is commonly used by couples.

**Multiple life insurance**: Benefits are paid out under various conditions (e.g., first death, second death, or the death of a specific person) and are commonly used to insure business partners.

**Universal Life Insurance**: A combination of life coverage and savings, where premiums earn interest. Flexible premiums can be skipped if the account balance covers costs, with the account value as the cash value, though early surrender may incur penalties

Type A UL: Death benefits are level

Type B UL: Death benefits increase as the invested premiums earn interest

**UWP Insurance**: Premiums are used to purchase units of a with-profit fund, similar to a unit trust, which can be invested in deposits, stocks, bonds, or real estate, and the fund value is represented by the bid value of units, may rise or fall with investment performance

**Equity-linked Insurance**:

**Equity-indexed Annuity**: The premium earns interest periodically based on a portion of an equity index (e.g., S&P 500), often with a cap rate, and includes a guaranteed minimum return of around 1%–3% after expenses.

**Variable Annuity**: Pays out the accumulated premium value at maturity or death—usually as a lump sum, though it may be converted to a life annuity; the benefit depends on investment performance, with various fund options, and guaranteed minimum death or maturity benefits are available to protect the initial investment

**SPIA**: The annuity benefit of a Single Premium Immediate Annuity (SPIA) begins immediately after the contract is written, with the policyholder paying a single premium at the start.

**SPDA**: The annuity benefit of a Single Premium Deferred Annuity (SPDA) begins at a future specified date, with the policyholder paying a single premium upfront.

**Guaranteed Annuity**: During the guarantee period, payments are made whether the annuitant survives or not. After that, payments are made only if the annuitant is alive. Annuity investors don't need underwriting, as the insurer faces the longevity risk

**Multiple Life Annuity**:

Joint Life Annuity: Payments stop after the first death of the couple.

Last Survivor Annuity: Payments stop after the second death of the couple.

Reversionary Annuity: Payments start after the first death and stop after the second death.

**Critical Illness Insurance**: Critical Illness Insurance pays a lump sum if the policyholder is diagnosed with a covered illness listed in the policy.

it often added as a rider to life insurance, accelerating all or part of the death benefit upon diagnosis. An extra premium is required, and the benefit can be used freely (indemnity design), such as for medical bills, mortgage payments, or income replacement

**Hospital Indemnity Insurance**: Hospital Indemnity Insurance provides fixed cash benefits for covered hospital stays or services, paid per day, week, visit, or event, regardless of actual medical costs.

**Activities of Daily Living**: Health professionals often use a person's inability to perform daily self-care activities as a measurement of their functional status

- Bathing
- Dressing (pull clothes on, fasten buttons, or close zippers)
- Eating (not including cooking or chewing and swallowing)
- Toileting (ability to use the toilet without help and manage personal hygiene)
- Continence (ability to control bladder and bowel functions)
- Functional mobility, often referred to as transferring or ambulating

**Chronic Illness Insurance**: A chronic illness is a long-term medical condition (lasting 3+ months), such as cancer, diabetes, or stroke, often requiring help with daily activities (ADLs)

it often added as a rider to life insurance, pays a lump sum or annuity when the policyholder is diagnosed and can no longer perform at least two ADLs. The benefit helps cover care costs not fully paid by health insurance.

**Long Term Care Insurance**: Covers ongoing support services for individuals needing daily care due to chronic illness, accidents, or cognitive decline. It can be purchased as a standalone policy or added as a rider.

**Disability Income Insurance**: provides supplementary income benefit in the event of an illness or accident resulting in a disability that prevents the insured from working at full capacity

**CCRCs**: Continuing Care Retirement Communities are residential facilities for seniors with 4 livign options: Independent living, Assisted living, Skilled nursing care, Memory support care

**Structured Settlements**: A negotiated arrangement where an injured party resolves a personal injury claim by receiving a lump sum and/or periodic payments over time from the responsible party.

**Pension**: A fixed sum to be paid regularly to a person; pension plans can be set up by employers, insurance companies, governments, or trade unions. Employer-sponsored plans often involve contributions from both the employer and the employee. Government pensions are usually funded through taxes and provide income during retirement or disability

**Defined Contribution Plan**: A retirement savings plan where both the employer and/or employee contribute a specified amount, usually a percentage of the employee's salary, up to a maximum limit.

**Defined Benefit Plan**: A retirement plan where the employer promises a specified annual benefit, calculated based on a formula that considers the employee's earnings history and years of service.

$$\text{Annual pension benefit } = (\text{ accrual rate }) \times (\text{ number of years of service }) \times S$$

where accrual rate is typically around 1% to 2%, and $S$ can be

(a) a final salary plan, which uses the final $m$-year average salary (3 5 years);

(b) a career average salary plan, which uses the average salary earned over the entire employment period;

(c) a career average revalued earnings (CARE) plan, which is similar to (b) but adjusts all salaries for inflation.

# A4.2 Survival Models and Estimation

## SURVIVAL FUNCTIONS

**Age-At-Death RV:** $T_0$ → The dying age for an individual currently at age 0.

**Future lifetime RV:** $T_x$ → The pdf of $T_x$ is $f_x(t) = {}_tp_x\ \mu_{x+t}$.

**Curtate future lifetime RV:** $K_x = \lfloor T_x \rfloor$ → The pmf of $K_x$ is $\Pr(K_x = k) = {}_kp_x\ q_{x+k}$.

**Survival function:** $S_x(t) = \Pr(T_x > t)$

**Formulas:**

$S_x(u+t) = S_x(u)\ S_{x+u}(t)$ → $S_{x+u}(t) = \dfrac{S_x(u+t)}{S_x(u)}$

$S_0(x+t) = S_0(x)\ S_x(t)$ → $S_x(t) = \dfrac{S_0(x+t)}{S_0(x)}$

**Three conditions :**

(1) $S_x(0) = 1$

(2) $\lim_{t\to\infty} S_x(t) = 0$

(3) $S_x(t)$ must be a non-increasing function of $t$.

## ACTUARIAL NOTATION

**Survival probability:** ${}_tp_x = S_x(t) = \Pr(T_x > t)$

**Mortality probability:** ${}_tq_x = 1 - {}_tp_x = F_x(t) = \Pr(T_x \le t)$

**Formulas:**

$${}_{t+u}p_x = {}_tp_x\ {}_up_{x+t}$$

$${}_{t+u}q_x = {}_tq_x + {}_tp_x\ {}_uq_{x+t}$$

$${}_{t|u}q_x = \Pr(t < T_x \le t+u) = {}_tp_x\ {}_uq_{x+t} = {}_tp_x - {}_{t+u}p_x = {}_{t+u}q_x - {}_tq_x$$

## FORCE OF MORTALITY

**Definition:** $\mu_x(t) = \mu_{x+t} = \frac{f_x(t)}{S_x(t)} = -\frac{d}{dt}\log S_x(t)$

**Condition:** $\lim_{t\to\infty} \int_0^t \mu_s ds = \infty$

**Formulas:**

$${}_tp_x = e^{-\int_0^t \mu_{x+s}ds} = e^{-\int_x^{x+t}\mu_s ds}$$

$${}_tq_x = \int_0^t {}_sp_x\ \mu_{x+s}ds$$

$${}_{t|u}q_x = \int_t^{t+u} {}_sp_x\ \mu_{x+s}ds$$

## EXPECTED FUTURE LIFETIME

**Expectations:**

$$E[T_x] = \overset{o}{e}_x = \int_0^\infty t\ {}_tp_x\ \mu_{x+t}dt = \int_0^\infty {}_tp_x dt$$

$$E[K_x] = e_x = \sum_{k=1}^{\infty} k\ {}_kp_x\ q_{x+k} = \sum_{k=1}^{\infty} {}_kp_x$$

$$E[\min(T_x, n)] = \overset{o}{e}_{x:\overline{n|}} = \int_0^n t\ {}_tp_x\ \mu_{x+t}dt + n\,{}_np_x = \int_0^n {}_tp_x dt$$

$$E[\min(K_x, n)] = e_{x:\overline{n|}} = \sum_{k=1}^{n-1} k\ {}_kp_x\ q_{x+k} + n\,{}_np_x = \sum_{k=1}^{n} {}_kp_x$$

**Second moments:**

$$E\left[T_x^2\right] = \int_0^\infty t^2\,{}_tp_x\ \mu_{x+t}dt = \int_0^\infty 2t\,{}_tp_x dt$$

$$E\left[K_x^2\right] = \sum_{k=1}^{\infty} k^2\,{}_kp_x\ q_{x+k} = \sum_{k=1}^{\infty} (2k-1)\,{}_kp_x = 2\sum_{k=1}^{\infty} k\ {}_kp_x - e_x$$

$$E\left[\min(T_x,n)^2\right] = \int_0^n t^2 {}_tp_x\ \mu_{x+t}dt + n^2 {}_np_x = \int_0^n 2t {}_tp_x dt$$

$$E\left[\min(K_x,n)^2\right] = \sum_{k=1}^{n-1} k^2 {}_kp_x\ q_{x+k} + n^2 {}_np_x = \sum_{k=1}^{n} (2k-1)\, {}_kp_x$$

**Variance:** $Var(T_x) = E\left[T_x^2\right] - E\left[T_x\right]^2$

$Var(K_x) = E\left[K_x^2\right] - E\left[K_x\right]^2$

**Recursive formulas:** $\overset{o}{e}_x = \overset{o}{e}_{x:\overline{n}|} + {}_np_x\ \overset{o}{e}_{x+n}$

$e_x = e_{x:\overline{n}|} + {}_np_x\ e_{x+n} \rightarrow e_x = p_x + p_x\ e_{x+1}$

$\overset{o}{e}_{x:\overline{n}|} = \overset{o}{e}_{x:\overline{m}|} + {}_mp_x\ \overset{o}{e}_{x+m:\overline{n-m}|}$ for $m < n$

$e_{x:\overline{n}|} = e_{x:\overline{m}|} + {}_mp_x\ e_{x+m:\overline{n-m}|}$ for $m < n$ $\rightarrow e_{x:\overline{n}|} = p_x + p_x\ e_{x+1:\overline{n-1}|}$

## MORTALITY LAWS

**Gompertz's law:** $\mu_x = Bc^x$ for $c > 1$ $\rightarrow {}_tp_x = e^{-\frac{Bc^x\left(c^t-1\right)}{\log c}}$

**Makeham's law:** $\mu_x = A + Bc^x$ for $c > 1$ $\rightarrow {}_tp_x = e^{-At-\frac{Bc^x\left(c^t-1\right)}{\log c}}$

**Weibull distribution:** $\mu_x = kx^n$ $\rightarrow {}_tp_x = e^{-\frac{k\left((x+t)^{n+1}-x^{n+1}\right)}{n+1}}$

**Exponential distribution:** (Constant force of mortality) $\mu_x = \mu$ $\rightarrow {}_tp_x = e^{-\mu t}$

**Uniform distribution:** (De Moivre's law) $\mu_x = \frac{1}{\omega-x}$ for $0 \le x \le \omega$ $\rightarrow {}_tp_x = 1 - \frac{t}{\omega-x}$

**Beta distribution:** (Generalized De Moivre) $\mu_x = \frac{\alpha}{\omega-x}$ for $0 \le x \le \omega$ $\rightarrow {}_tp_x = \left(1 - \frac{t}{\omega-x}\right)^\alpha$

## LIFE TABLES

**Number of lives:** $l_x$

**Number of deaths:** ${}_td_x = l_x - l_{x+t}$

**Formulas:** ${}_tp_x = \dfrac{l_{x+t}}{l_x}$ ${}_tq_x = \dfrac{{}_td_x}{l_x} = \dfrac{l_x - l_{x+t}}{l_x}$ ${}_{t|u}q_x = \dfrac{l_{x+t} - l_{x+t+u}}{l_x}$

***k*-year select period:**

| | | |
|---|---|---|
| for $h < k$ | $q_{[x]+h} < q_{x+h}$ | $p_{[x]+h} > p_{x+h}$ |
| for $h \ge k$ | $q_{[x]+h} = q_{x+h}$ | $p_{[x]+h} = p_{x+h}$ |

## FRACTIONAL AGE ASSUMPTIONS

**UDD between integral ages:** $l_{x+s} = l_x - sd_x$ $\rightarrow {}_sq_x = sq_x$ ${}_sq_{x+t} = \dfrac{Sq_x}{1-tq_x}$ $q_x = {}_sp_x\ \mu_{x+s}$

**CFM between integral ages:** $l_{x+s} = l_x \times (p_x)^s$ $\rightarrow {}_sp_x = (p_x)^s$ ${}_sp_{x+t} = (p_x)^s$ $\mu_{x+s} = -\log p_x$

These are for $0 \le s, t \le 1$ and $0 \le s+t \le 1$.

## EMPIRICAL ESTIMATION

**For seriatim data:** $\hat{S}(t) = \dfrac{\text{Number of survivors at time } t}{n} = \dfrac{n_t}{n}$

**Variance:** $\text{Var}(\hat{S}(t)) \approx \dfrac{n_t\,(n-n_t)}{n^3} = \hat{S}(t)^2\left(\frac{1}{n_t} - \frac{1}{n}\right)$

**For grouped data:** $\hat{S}(t) = \dfrac{(t_U - t)\,\hat{S}(t_L) + (t - t_L)\,\hat{S}(t_U)}{t_U - t_L}$ This is called ogive empirical survival function.

**Where:** $t_L \le t < t_U$

## KM AND NA ESTIMATES

**Define the following:**

$t_{(j)}$ is the time of each event for a mortality study — An event can be an entry, exit, or death.

$c_j^L$ is the number of entries at time $t_{(j)}$ — These are left truncated observations.

$c_j^R$ is the number of exits at time $t_{(j)}$ — These are right censored observations.

$d_j$ is the number of deaths at time $t_{(j)}$ — They are observations with exact values.

$r_j = r_{j-1} + c_{j-1}^L - c_{j-1}^R - d_{j-1}$ is the number of (active) lives at time $t_{(j)}$

**Kaplan-Meier estimator:** $\hat{S}(t) = \prod_{j:t_{(j)} \le t} \left(1 - \frac{d_j}{r_j}\right)$ Also called product limit estimator.

**Greenwood's formula:** $\mathrm{Var}(\hat{S}(t)) \approx (\hat{S}(t))^2 \sum_{j:t_{(j)} \le t} \left(\frac{d_j}{r_j(r_j - d_j)}\right)$

**Linear CI for $S(t)$:** $\hat{S}(t) \pm z\sqrt{\mathrm{Var}(\hat{S}(t))}$

**Log-transformed CI for $S(t)$:** $\left(\hat{S}(t)^{\frac{1}{U}},\ \hat{S}(t)^{U}\right)$ where $U = \exp\left(\dfrac{z\sqrt{\widehat{Var}\left(\hat{S}(t)\right)}}{\hat{S}(t)\log \hat{S}(t)}\right)$

**Nelson-Aalen estimator:** $\widehat{H}(t) = \sum_{j:t_{(j)} \le t} \left(\frac{d_j}{r_j}\right) \rightarrow \hat{S}(t) = e^{-\hat{H}(t)}$

**Klein's formula:** $\mathrm{Var}(\widehat{H}(t)) \approx \sum_{j:t_{(j)} \le t} \left(\frac{d_j(r_j - d_j)}{r_j^3}\right) \rightarrow \mathrm{Var}(\hat{S}(t)) \approx (\hat{S}(t))^2 \sum_{j:t_{(j)} \le t} \left(\frac{d_j(r_j - d_j)}{r_j^3}\right)$

**Linear CI for $H(t)$:** $\widehat{H}(t) \pm z\sqrt{\mathrm{Var}(\widehat{H}(t))}$

**Log-transformed CI for $H(t)$:** $\left(\frac{\hat{H}(t)}{U},\ \hat{H}(t)\,U\right)$ where $U = \exp\left(\dfrac{z\sqrt{\widehat{Var}\left(\hat{H}(t)\right)}}{\hat{H}(t)}\right)$

**Exponential extrapolation:** $\hat{S}(t) = \hat{S}(t_{\max})^{t/t_{\max}}$ for $t \ge t_{\max}$

**Where:** $t_{\max}$ is the time the study ends or it is the time of the last event.

# Multi-State Model

## MARKOV CHAIN PROBABILITIES

**Transition probability:**

${}_tp_x^{ij}$ is the probability that a life aged $x$ who is currently in state $i$ will be in state $j$ at time $t$.

${}_tp_x^{ii}$ is the probability that a life aged $x$ who is currently in state $i$ will be in state $i$ at time $t$.

${}_tp_x^{\overline{ii}}$ is the probability that a life aged $x$ who is currently in state $i$ will be in state $i$ until time $t$.

${}_tp_x^{\overline{ii}} \le {}_tp_x^{ii}$

$${}_0p_x^{ij} = \begin{cases} 1, & i = j \\ 0, & i \ne j \end{cases}$$

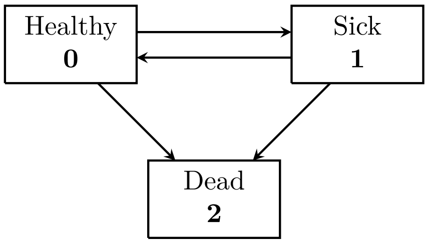
**Chapman-Kolmogorov equations**: $${}_kp^{ij}_x = \sum_s {}_rp^{is}_x \; {}_{k-r}p^{sj}_{x+r}$$

**Forces of transition**: $$\mu^{ij}_{x+t} \quad \rightarrow \quad {}_tp^{\overline{ii}}_x = e^{-\int_0^t \sum_{j\neq i} \mu^{ij}_{x+s} ds}$$

**Kolmogorov's forward equations**: $$\frac{d}{dt}{}_tp^{ij}_x = \frac{{}_{t+h}p^{ij}_x - {}_tp^{ij}_x}{h} = \sum_{k\neq j}\left({}_tp^{ik}_x \; \mu^{kj}_{x+t} - {}_tp^{ij}_x \; \mu^{jk}_{x+t}\right)$$

## DISABILITY INCOME INSURANCE MODEL

**Discrete DII model**:




**Transition matrix**: $$\mathbf{P}^{(t)} = \begin{pmatrix} p^{00}_{x+t} & p^{01}_{x+t} & p^{02}_{x+t} \\ p^{10}_{x+t} & p^{11}_{x+t} & p^{12}_{x+t} \\ p^{20}_{x+t} & p^{21}_{x+t} & p^{22}_{x+t} \end{pmatrix} = \begin{pmatrix} p^{00}_{x+t} & p^{01}_{x+t} & p^{02}_{x+t} \\ p^{10}_{x+t} & p^{11}_{x+t} & p^{12}_{x+t} \\ 0 & 0 & 1 \end{pmatrix}$$

**Transition Probabilities**:

$${}_2p^{00}_x = p^{00}_x p^{00}_{x+1} + p^{01}_x p^{10}_{x+1} + p^{02}_x p^{20}_{x+1}$$

$${}_2p^{\overline{00}}_x = p^{00}_x p^{00}_{x+1}$$

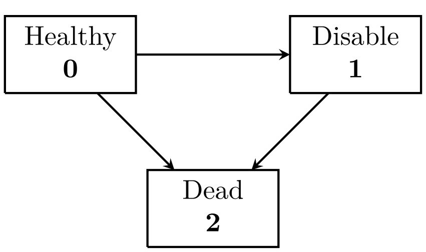
$${}_2p^{10}_x = \begin{pmatrix} p^{10}_x & p^{11}_x & p^{12}_x \end{pmatrix} \begin{pmatrix} p^{00}_{x+1} \\ p^{10}_{x+1} \\ 0 \end{pmatrix}$$

$${}_3p^{10}_x = \begin{pmatrix} p^{10}_x & p^{11}_x & p^{12}_x \end{pmatrix} \begin{pmatrix} p^{00}_{x+1} & p^{01}_{x+1} & p^{02}_{x+1} \\ p^{10}_{x+1} & p^{11}_{x+1} & p^{12}_{x+1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^{00}_{x+2} \\ p^{10}_{x+2} \\ 0 \end{pmatrix}$$

$${}_4p^{10}_x = \begin{pmatrix} p^{10}_x & p^{11}_x & p^{12}_x \end{pmatrix} \begin{pmatrix} p^{00}_{x+1} & p^{01}_{x+1} & p^{02}_{x+1} \\ p^{10}_{x+1} & p^{11}_{x+1} & p^{12}_{x+1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^{00}_{x+2} & p^{01}_{x+2} & p^{02}_{x+2} \\ p^{10}_{x+2} & p^{11}_{x+2} & p^{12}_{x+2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p^{00}_{x+3} \\ p^{10}_{x+3} \\ 0 \end{pmatrix}$$

## PERMANENT DISABILITY MODEL

**Continuous PD model**:




**Transition Intensities**: $\mu^{01}_{x+t}$ $\quad$ $\mu^{02}_{x+t}$ $\quad$ $\mu^{12}_{x+t}$

**Transition Probabilities**:

$${}_tp^{00}_x = e^{-\int_0^t (\mu^{01}_{x+s}+\mu^{02}_{x+s})ds} \qquad {}_tp^{01}_x = \int_0^t {}_sp^{00}_x \; \mu^{01}_{x+s} \; {}_{t-s}p^{11}_x \, ds \qquad {}_tp^{02}_x = 1 - {}_tp^{00}_x - {}_tp^{01}_x$$

$${}_tp^{10}_x = 0 \qquad {}_tp^{11}_x = e^{-\int_0^t \mu^{12}_{x+s} ds} \qquad {}_tp^{12}_x = 1 - {}_tp^{11}_x$$

$${}_tp^{20}_x = 0 \qquad {}_tp_x{}^{21} = 0 \qquad {}_tp_x{}^{22} = 1$$

## MULTIPLE DECREMENT PROBABILITIES

**Forces of decrement**: $\mu_{x+s}^{(\tau)} = \sum_j \mu_{x+s}^{(j)} \rightarrow {}_tp_x^{(\tau)} = e^{-\int_0^t \mu_{x+s}^{(\tau)} ds}$

$\rightarrow {}_tq_x^{(\tau)} = \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(\tau)} ds \rightarrow \mu_{x+t}^{(\tau)} = \frac{\frac{d}{dt}\, {}_tq_x^{(\tau)}}{{}_tp_x^{(\tau)}}$

**Multiple decrement probabilities**: ${}_tp_x^{(j)}$ does not exist.

${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(j)} ds \rightarrow {}_tq_x^{(\tau)} = \sum_j {}_tq_x^{(j)} \rightarrow \mu_{x+t}^{(j)} = \frac{\frac{d}{dt}\, {}_tq_x^{(j)}}{{}_tp_x^{(\tau)}}$

**Associated single decrement probabilities**:

${}_tp_x^{\prime(j)} = e^{-\int_0^t \mu_{x+s}^{(j)} ds} \rightarrow {}_tp_x^{(\tau)} = \prod_j {}_tp_x^{(j)}$

${}_tq_x^{\prime(j)} = \int_0^t {}_sp_x^{\prime(j)} \mu_{x+s}^{(j)} ds \rightarrow \mu_{x+t}^{(j)} = \frac{\frac{d}{dt}\, {}_tq_x^{\prime(j)}}{{}_tp_x^{\prime(j)}}$

**Formulas**: ${}_{t+u}p_x^{(\tau)} = {}_tp_x^{(\tau)}\, {}_up_{x+t}^{(\tau)}$

${}_{t+u}q_x^{(\tau)} = {}_tq_x^{(\tau)} + {}_tp_x^{(\tau)}\, {}_uq_{x+t}^{(\tau)}$

${}_{t+u}q_x^{(j)} = {}_tq_x^{(j)} + {}_tp_x^{(\tau)}\, {}_uq_{x+t}^{(j)}$

${}_{t|u}q_x^{(j)} = {}_tp_x^{(\tau)}\, {}_uq_{x+t}^{(j)}$

**Multiple decrement table**: $l_{x+t}^{(\tau)} = l_x^{(\tau)}\ {}_tp_x^{(\tau)}$

$\rightarrow d_{x+t}^{(\tau)} = l_x^{(\tau)}\, {}_tp_x^{(\tau)}\, q_{x+t}^{(\tau)}$

$\rightarrow d_{x+t}^{(j)} = l_x^{(\tau)}\, {}_tp_x^{(\tau)}\, q_{x+t}^{(j)}$

# A4.3 Present Value Random Variables

## Insurance

### ACTUARIAL FUNCTIONS

**Endowment:** ${}_nE_x = A_{x:\overset{\,1}{\overline{n|}}} = v^n\, {}_np_x$   ${}_n^2E_x = {}^2A_{x:\overset{\,1}{\overline{n|}}} = (v^n)^2\, {}_np_x$

**Insurance (Continuous):**

$\bar{A}_x = \int_0^\infty v^t\, {}_tp_x\ \mu_{x+t} dt$   ${}^2\bar{A}_x = \int_0^\infty (v^t)^2\ {}_tp_x\ \mu_{x+t} dt$

$\bar{A}_{\overset{1}{x}:\overline{n|}} = \int_0^n v^t\, {}_tp_x\ \mu_{x+t} dt$   ${}^2\bar{A}_{\overset{1}{x}:\overline{n|}} = \int_0^n (v^t)^2\ {}_tp_x\ \mu_{x+t} dt$

$\bar{A}_{x:\overline{n|}} = \int_0^n v^t\, {}_tp_x\ \mu_{x+t} dt + v^n\, {}_np_x$   ${}^2\bar{A}_{x:\overline{n|}} = \int_0^n (v^t)^2\ {}_tp_x\ \mu_{x+t} dt + (v^n)^2\ {}_np_x$

**Insurance (Discrete):**

$A_x = \sum_{k=0}^{\infty} v^{k+1}\, {}_kp_x\ q_{x+k}$   ${}^2A_x = \sum_{k=0}^{\infty} \left(v^{k+1}\right)^2\ {}_kp_x\ q_{x+k}$

$A_{\overset{1}{x}:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\ q_{x+k}$   ${}^2A_{\overset{1}{x}:\overline{n|}} = \sum_{k=0}^{n-1} \left(v^{k+1}\right)^2\ {}_kp_x\ q_{x+k}$

$A_{x:\overline{n|}} = \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x\ q_{x+k} + v^n\, {}_np_x$   ${}^2A_{x:\overline{n|}} = \sum_{k=0}^{n-1} \left(v^{k+1}\right)^2\ {}_kp_x\ q_{x+k} + (v^n)^2\ {}_np_x$

**Insurance (mthly):** $A_{\overset{1}{x}:\overline{n|}}^{(m)} = \sum_{k=0}^{nm-1} v^{k/m+1/m}\ {}_{k/m}p_x\ {}_{1/m}q_{x+k/m}$

**Relations:**

$\bar{A}_x = \bar{A}_{\overset{1}{x}:\overline{n|}} + {}_nE_x\ \bar{A}_{x+n}$   $A_x = A_{\overset{1}{x}:\overline{n|}} + {}_nE_x\ A_{x+n}$

$\bar{A}_{x:\overline{n|}} = \bar{A}_{\overset{1}{x}:\overline{n|}} + {}_nE_x$   $A_{x:\overline{n|}} = A_{\overset{1}{x}:\overline{n|}} + {}_nE_x$

${}_{n|}\bar{A}_x = {}_nE_x\ \bar{A}_{x+n}$   ${}_{n|}A_x = {}_nE_x A_{x+n}$

**Recursive formulas:**

$A_x = vq_x + vp_x\ A_{x+1}$   ${}^2A_x = v^2q_x + v^2p_x\ {}^2A_{x+1}$

$A_{\overset{1}{x}:\overline{n|}} = vq_x + vp_x\ A_{\overset{\,1}{x+1}:\overline{n-1|}}$   ${}^2A_{\overset{1}{x}:\overline{n|}} = v^2q_x + v^2p_x\ {}^2A_{\overset{\,1}{x+1}:\overline{n-1|}}$

## PRESENT VALUES

| Policy | Death benefit paid at the moment of death | Death benefit paid at the end of the year of death |
|---|---|---|
| Whole life insurance | $Z = bv^{T_x}, T_x > 0$ | $Z = bv^{K_x+1}, \quad K_x = 0,1,2,\dots,\infty$ |
| n-year term insurance | $Z = \begin{cases} bv^{T_x}, & T_x < n \\ 0, & T_x \geq n \end{cases}$ | $Z = \begin{cases} bv^{K_x+1}, & K_x = 0,1,2,\dots,n-1 \\ 0, & K_x = n, n+1,\dots,\infty \end{cases}$ |
| n-year endowment insurance | $Z = \begin{cases} bv^{T_x}, & T_x < n \\ bv^n, & T_x \geq n \end{cases}$ | $Z = \begin{cases} bv^{K_x+1}, & K_x = 0,1,2,\dots,n-1 \\ bv^n, & K_x = n, n+1,\dots,\infty \end{cases}$ |
| n-year pure endowment | $Z = \begin{cases} 0, & T_x < n \\ bv^n, & T_x \geq n \end{cases}$ | |
| n-year deferred whole life insurance | $Z = \begin{cases} 0, & T_x < n \\ bv^{T_x}, & T_x \geq n \end{cases}$ | $Z = \begin{cases} 0, & K_x = 0,1,2,\dots,n-1 \\ bv^{K_x+1}, & K_x = n, n+1,\dots,\infty \end{cases}$ |

## EXPECTED PRESENT VALUES

| Policy | Death benefit paid at the moment of death | Death benefit paid at the end of the year of death |
|---|---|---|
| Whole life insurance | $E[Z] = b\bar{A}_x$ | $E[Z] = bA_x$ |
| n-year term insurance | $E[Z] = b\bar{A}^{\,1}_{x:\overline{n}\vert}$ | $E[Z] = bA^{\,1}_{x:\overline{n}\vert}$ |
| n-year endowment insurance | $E[Z] = b\bar{A}_{x:\overline{n}\vert}$ | $E[Z] = bA_{x:\overline{n}\vert}$ |
| n-year pure endowment | $E[Z] = b\,{}_nE_x$ | |
| n-year deferred whole life insurance | $E[Z] = b\,{}_{n\vert}\bar{A}_x$ | $E[Z] = b\,{}_{n\vert}A_x$ |

## VARIANCE OF PRESENT VALUES

| Policy | Death benefit paid at the moment of death | Death benefit paid at the end of the year of death |
|---|---|---|
| Whole life insurance | $Var(Z) = b^2\left({}^2\bar{A}_x - (\bar{A}_x)^2\right)$ | $Var(Z) = b^2\left({}^2A_x - (A_x)^2\right)$ |
| n-year term insurance | $Var(Z) = b^2\left({}^2\bar{A}^{\,1}_{x:\overline{n}\vert} - \left(\bar{A}^{\,1}_{x:\overline{n}\vert}\right)^2\right)$ | $Var(Z) = b^2\left({}^2A^{\,1}_{x:\overline{n}\vert} - \left(A^{\,1}_{x:\overline{n}\vert}\right)^2\right)$ |
| n-year endowment insurance | $Var(Z) = b^2\left({}^2\bar{A}_{x:\overline{n}\vert} - (\bar{A}_{x:\overline{n}\vert})^2\right)$ | $Var(Z) = b^2\left({}^2A_{x:\overline{n}\vert} - (A_{x:\overline{n}\vert})^2\right)$ |
| n-year pure endowment | $Var(Z) = b^2\left({}^2_nE_x - ({}_nE_x)^2\right)$ | |
| n-year deferred whole life insurance | $Var(Z) = b^2\left({}^2_{n\vert}\bar{A}_x - ({}_{n\vert}\bar{A}_x)^2\right)$ | $Var(Z) = b^2\left({}^2_{n\vert}A_x - ({}_{n\vert}A_x)^2\right)$ |

## APPROXIMATIONS

**UDD between integral ages:**

$\bar{A}_x = \frac{i}{\delta}A_x$ $\quad$ $\bar{A}^{\,1}_{x:\overline{n}\vert} = \frac{i}{\delta}A^{\,1}_{x:\overline{n}\vert}$ $\quad$ $\bar{A}_{x:\overline{n}\vert} \neq \frac{i}{\delta}A_{x:\overline{n}\vert}$.

$A^{(m)}_x = \frac{i}{i^{(m)}}A_x$

${}^2\bar{A}_x = \frac{2i+i^2}{2\delta}\,{}^2A_x$

**Claims acceleration approach:** $\bar{A}_x = (1+i)^{0.5}\ A_x$ $\qquad \bar{A}^{\,1}_{x:\overline{n}|} = (1+i)^{0.5}\ A^{\,1}_{x:\overline{n}|}$ $\qquad \bar{A}_{x:\overline{n}|} \neq (1+i)^{0.5}\ A_{x:\overline{n}|}.$

$A_x^{(m)} = (1+i)^{\frac{m-1}{2m}}\ A_x$

${}^2\bar{A}_x = (1+i)\ {}^2A_x$

# Annuities

## ACTUARIAL FUNCTIONS

**Annuity (Continuous):**

$\bar{a}_x = \int_0^\infty \left(\frac{1-v^t}{\delta}\right) {}_tp_x\ \mu_{x+t}dt$ $\qquad {}^2\bar{a}_x = \int_0^\infty \left(\frac{1-v^{2t}}{2\delta}\right) {}_tp_x\ \mu_{x+t}dt$

$\bar{a}_{x:\overline{n}|} = \int_0^n \left(\frac{1-v^t}{\delta}\right) {}_tp_x\ \mu_{x+t}dt + \left(\frac{1-v^n}{\delta}\right) {}_np_x$ $\qquad {}^2\bar{a}_{x:\overline{n}|} = \int_0^n \left(\frac{1-v^{2t}}{2\delta}\right) {}_tp_x\ \mu_{x+t}dt + \left(\frac{1-v^{2n}}{2\delta}\right) {}_np_x$

**Annuity (Discrete):**

$\ddot{a}_x = \sum_{k=0}^{\infty} \left(\frac{1-v^{k+1}}{d}\right) {}_kp_x\ q_{x+k}$ $\qquad {}^2\ddot{a}_x = \sum_{k=0}^{\infty} \left(\frac{1-v^{2k+2}}{2d-d^2}\right) {}_kp_x\ q_{x+k}$

$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \left(\frac{1-v^{k+1}}{d}\right) {}_kp_x\ q_{x+k} + \left(\frac{1-v^n}{d}\right) {}_np_x$ $\qquad {}^2\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \left(\frac{1-v^{2k+2}}{2d-d^2}\right) {}_kp_x\ q_{x+k} + \left(\frac{1-v^{2n}}{2d-d^2}\right) {}_np_x$

**Simpler formulas:**

$\bar{a}_x = \int_0^\infty v^t\,{}_tp_x dt$ $\qquad {}^2\bar{a}_x = \int_0^\infty v^{2t}\,{}_tp_x dt$

$\bar{a}_{x:\overline{n}|} = \int_0^n v^t\,{}_tp_x dt$ $\qquad {}^2\bar{a}_{x:\overline{n}|} = \int_0^n v^{2t}\,{}_tp_x dt$

$\ddot{a}_x = \sum_{k=0}^{\infty} v^k\,{}_kp_x$ $\qquad {}^2\ddot{a}_x = \sum_{k=0}^{\infty} v^{2k}\,{}_kp_x$

$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k\,{}_kp_x$ $\qquad {}^2\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{2k}\,{}_kp_x$

$\ddot{a}^{(m)}_{x:\overline{n}|} = \sum_{k=0}^{nm-1} \frac{1}{m} v^{k/m}\,{}_{k/m}p_x$

**Relations:**

$\bar{a}_x = \bar{a}_{x:\overline{n}|} + {}_nE_x\ \bar{a}_{x+n}$ $\qquad \ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x\ \ddot{a}_{x+n}$

${}_{n|}\bar{a}_x = {}_nE_x\ \bar{a}_{x+n}$ $\qquad {}_{n|}\ddot{a}_x = {}_nE_x\ \ddot{a}_{x+n}$

$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + {}_nE_x\ \bar{a}_{x+n}$ $\qquad \ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_nE_x\ \ddot{a}_{x+n}$

$\ddot{a}_x = 1 + a_x$

$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|} = 1 + a_{x:\overline{n}|} - {}_nE_x$

**Recursive formulas:**

$\ddot{a}_x = 1 + vp_x\ \ddot{a}_{x+1}$ $\qquad {}^2\ddot{a}_x = 1 + v^2p_x\ {}^2\ddot{a}_{x+1}$

$\ddot{a}_{x:\overline{n}|} = 1 + vp_x\ \ddot{a}_{x+1:\overline{n-1}|}$ $\qquad {}^2\ddot{a}_{x:\overline{n}|} = 1 + v^2p_x\ {}^2\ddot{a}_{x+1:\overline{n-1}|}$

**Insurance to annuity:**

$\bar{a}_x = \frac{1-\bar{A}_x}{\delta}$ $\qquad {}^2\bar{a}_x = \frac{1-{}^2\bar{A}_x}{2\delta}$

$\bar{a}_{x:\overline{n}|} = \frac{1-\bar{A}_{x:\overline{n}|}}{\delta}$ $\qquad {}^2\bar{a}_{x:\overline{n}|} = \frac{1-{}^2\bar{A}_{x:\overline{n}|}}{2\delta}$

$\ddot{a}_x = \frac{1-A_x}{d}$ $\qquad {}^2\ddot{a}_x = \frac{1-{}^2A_x}{2d-d^2}$

$\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$ $\qquad {}^2\ddot{a}_{x:\overline{n}|} = \frac{1-{}^2A_{x:\overline{n}|}}{2d-d^2}$

$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$

## PRESENT VALUES

| Policy | Payments made continuously | Payments at the beginning of each year |
|---|---|---|
| Whole life annuity | $Y = b\left(\frac{1-v^{T_x}}{\delta}\right), T_x > 0$ | $Y = b\left(\frac{1-v^{K_x+1}}{d}\right), \quad K_x = 0,1,2,\ldots,\infty$ |
| n-year term annuity | $Y = \begin{cases} b\left(\frac{1-v^{T_x}}{\delta}\right), & T_x < n \\ b\left(\frac{1-v^n}{\delta}\right), & T_x \geq n \end{cases}$ | $Y = \begin{cases} b\left(\frac{1-v^{K_x+1}}{d}\right), & K_x = 0,1,2,\ldots,n-1 \\ b\left(\frac{1-v^n}{d}\right), & K_x = n,n+1,\ldots,\infty \end{cases}$ |
| n-year certain whole life annuity | $Y = \begin{cases} b\left(\frac{1-v^n}{\delta}\right), & T_x < n \\ b\left(\frac{1-v^{T_x}}{\delta}\right), & T_x \geq n \end{cases}$ | $Y = \begin{cases} b\left(\frac{1-v^n}{d}\right), & K_x = 0,1,2,\ldots,n-1 \\ b\left(\frac{1-v^{K_x+1}}{d}\right), & K_x = n,n+1,\ldots,\infty \end{cases}$ |
| n-year deferred whole life annuity | $Y = \begin{cases} 0, & T_x < n \\ b\left(\frac{v^n-v^{T_x}}{\delta}\right), & T_x \geq n \end{cases}$ | $Y = \begin{cases} 0, & K_x = 0,1,2,\ldots,n-1 \\ b\left(\frac{v^n-v^{K_x+1}}{d}\right), & K_x = n,n+1,\ldots,\infty \end{cases}$ |

## EXPECTED PRESENT VALUES

| Policy | Payments made continuously | Payments at the beginning of each year |
|---|---|---|
| Whole life annuity | $E\left[Y\right] = b\bar{a}_x$ | $E\left[Y\right] = b\ddot{a}_x$ |
| n-year term annuity | $E\left[Y\right] = b\bar{a}_{x:\overline{n}\rvert}$ | $E\left[Y\right] = b\ddot{a}_{x:\overline{n}\rvert}$ |
| n-year certain whole life annuity | $E\left[Y\right] = b\bar{a}_{\overline{x:\overline{n}\rvert}}$ | $E\left[Y\right] = b\ddot{a}_{\overline{x:\overline{n}\rvert}}$ |
| n-year deferred whole life annuity | $E\left[Y\right] = b_{n\rvert}\bar{a}_x$ | $E\left[Y\right] = b_{n\rvert}\ddot{a}_x$ |

## VARIANCE OF PRESENT VALUES

| Policy | Payments made continuously | Payments at the beginning of each year |
|---|---|---|
| Whole life annuity | $Var\left(Y\right) = b^2\frac{{}^2\bar{A}_x-\left(\bar{A}_x\right)^2}{\delta^2}$ | $Var\left(Y\right) = b^2\frac{{}^2A_x-\left(A_x\right)^2}{d^2}$ |
| n-year term annuity | $Var\left(Y\right) = b^2\frac{{}^2\bar{A}_{x:\overline{n}\rvert}-\left(\bar{A}_{x:\overline{n}\rvert}\right)^2}{\delta^2}$ | $Var\left(Y\right) = b^2\frac{{}^2A_{x:\overline{n}\rvert}-\left(A_{x:\overline{n}\rvert}\right)^2}{d^2}$ |

## APPROXIMATIONS

**UDD between integral ages:**

$$\bar{a}_x = \frac{id}{\delta^2}\ddot{a}_x - \frac{i-\delta}{\delta^2}$$

$$\ddot{a}_x^{(m)} = \frac{id}{i^{(m)}d^{(m)}}\ddot{a}_x - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}$$

$$\ddot{a}_{x:\overline{n}\rvert}^{(m)} = \frac{id}{i^{(m)}d^{(m)}}\ddot{a}_{x:\overline{n}\rvert} - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}\left(1-{}_nE_x\right)$$

**Woolhouse formula, 2 terms:**

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2}$$

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$$

$$\ddot{a}_{x:\overline{n}\rvert}^{(m)} \approx \ddot{a}_{x:\overline{n}\rvert} - \frac{m-1}{2m}\left(1-{}_nE_x\right)$$

**Woolhouse formula, 3 terms:**

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}\left(\mu_x+\delta\right)$$

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left(\mu_x+\delta\right)$$ where $\mu_x \approx -\frac{1}{2}\left(\log p_{x-1}+\log p_x\right)$ if not given.

$$\ddot{a}_{x:\overline{n}\rvert}^{(m)} \approx \ddot{a}_{x:\overline{n}\rvert} - \frac{m-1}{2m}\left(1-{}_nE_x\right) - \frac{m^2-1}{12m^2}\left(\mu_x+\delta-{}_nE_x\left(\mu_{x+n}+\delta\right)\right)$$

# A4.4 Long-Term Insurance Premium Calculation

## ACTUARIAL FUNCTIONS

**Premium (Discrete):** $P_x = \frac{A_x}{\ddot{a}_x}$ $\quad P_{\overset{1}{x}:\overline{n}|} = \frac{A_{\overset{1}{x}:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$ $\quad P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$

**Premium (Continuous):** $\bar{P}\left(\bar{A}_x\right) = \frac{\bar{A}_x}{\bar{a}_x}$ $\quad \bar{P}\left(\bar{A}_{\overset{1}{x}:\overline{n}|}\right) = \frac{\bar{A}_{\overset{1}{x}:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$ $\quad \bar{P}\left(\bar{A}_{x:\overline{n}|}\right) = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$

## EQUIVALENCE PRINCIPLE

**Loss-at-issue:** $_0L^n = PV(\text{Benefits}) - PV(\text{Premiums})$

$_0L^g = PV(\text{Benefits}) + PV(\text{Expenses}) - PV(\text{Premiums})$

**Equivalence principle:** $\mathrm{E}[_0L] = EPV(\text{Benefits}) - EPV(\text{Premiums}) = 0$

$\mathrm{E}[_0L^g] = EPV(\text{Benefits}) + EPV(\text{Expenses}) - EPV(\text{Premiums}) = 0$

## LOSS-AT-ISSUE RANDOM VARIABLE

| Fully continuous whole life insurance | Fully discrete whole life insurance |
|---|---|
| $_0L^g = (b+E)\,v^{T_x} - (G-e)\left(\frac{1-v^{T_x}}{\delta}\right), \quad T_x \geq 0$ <br> $_0L^g = \left(b+E+\frac{G-e}{\delta}\right)Z - \frac{G-e}{\delta}$ <br> $E\left[_0L^g\right] = (b+E)\,\bar{A}_x - (G-e)\,\bar{a}_x$ <br> $Var\left(_0L^g\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right)$ | $_0L^g = (b+E)\,v^{K_x+1} - (G-e)\left(\frac{1-v^{K_x+1}}{d}\right), \quad K_x = 0,1,2,\ldots,\infty$ <br> $_0L^g = \left(b+E+\frac{G-e}{d}\right)Z - \frac{G-e}{d}$ <br> $E\left[_0L^g\right] = (b+E)\,A_x - (G-e)\,\ddot{a}_x$ <br> $Var\left(_0L^g\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_x - \left(A_x\right)^2\right)$ |
| **Fully continuous n-year endowment insurance** | **Fully discrete n-year endowment insurance** |
| $_0L^g = \begin{cases}(b+E)\,v^{T_x} - (G-e)\left(\frac{1-v^{T_x}}{\delta}\right), & T_x < n \\ (b+E)\,v^n - (G-e)\left(\frac{1-v^n}{\delta}\right), & T_x \geq n\end{cases}$ <br> $_0L^g = \left(b+E+\frac{G-e}{\delta}\right)Z - \frac{G-e}{\delta}$ <br> $E\left[_0L^g\right] = (b+E)\,\bar{A}_{x:\overline{n}|} - (G-e)\,\bar{a}_{x:\overline{n}|}$ <br> $Var\left(_0L^g\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2\right)$ | $_0L^g = \begin{cases}(b+E)\,v^{K_x+1} - (G-e)\left(\frac{1-v^{K_x+1}}{d}\right), & K_x = 0,1,\ldots,n-1 \\ (b+E)\,v^n - (G-e)\left(\frac{1-v^n}{d}\right), & K_x = n,n+1,\ldots,\infty\end{cases}$ <br> $_0L^g = \left(b+E+\frac{G-e}{d}\right)Z - \frac{G-e}{d}$ <br> $E\left[_0L^g\right] = (b+E)\,A_{x:\overline{n}|} - (G-e)\,\ddot{a}_{x:\overline{n}|}$ <br> $Var\left(_0L^g\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right)$ |

## PORTFOLIO PERCENTILE PRINCIPLE

**Aggregate losses:** $S = L_1^g + L_2^g + \cdots + L_n^g \dot{\sim} N\left(\mu, \sigma^2\right)$

**Mean:** $\mu = E[S] = nE\left[_0L^g\right]$

**Variance:** $\sigma^2 = \mathrm{Var}(S) = n\,\mathrm{Var}\left(_0L^g\right)$

**Find premium such that:** $\Pr(S<0) \approx N\left(\frac{0-nE[_0L^g]}{\sqrt{n\,\mathrm{Var}(_0L^g)}}\right) = \alpha \quad \rightarrow \quad \frac{0-nE[_0L^g]}{\sqrt{n\,\mathrm{Var}(_0L^g)}} = z_\alpha$

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# A4.5 Long-Term Insurance Reserves

## PROSPECTIVE FORMULA

**Net Premium Policy Value:** $_tV^n = E\left[{}_tL \mid T_x \geq t\right] = EPV_t\text{ (Benefits)} - EPV_t\text{ (Net Premiums)}$

**Gross Premium Policy Value:** $_tV^g = E\left[{}_tL^g \mid T_x \geq t\right] = EPV_t\text{ (Benefits)} + EPV_t\text{ (Expenses)} - EPV_t\text{ (Gross Premiums)}$

**Expense Policy Value:** $_tV^e = {}_tV^g - {}_tV^n = EPV_t\text{(Expenses)} - EPV_t\text{(Expense Loadings)}$

## EXPECTATION AND VARIANCE

| Fully continuous whole life insurance | Fully discrete whole life insurance |
|---|---|
| $_tL^g = (b+E)\,v^{T_{x+t}} - (G-e)\left(\frac{1-v^{T_{x+t}}}{\delta}\right),\quad T_{x+t} \geq 0$ | $_kL^g = (b+E)\,v^{K_{x+k}+1} - (G-e)\left(\frac{1-v^{K_{x+k}+1}}{d}\right),$ $K_{x+k} = 0, 1, 2, \ldots, \infty$ |
| $_tL^g = \left(b+E+\frac{G-e}{\delta}\right)Z - \frac{G-e}{\delta}$ <br> $E\left[{}_tL^g \mid T_x \geq t\right] = (b+E)\bar{A}_{x+t} - (G-e)\bar{a}_{x+t}$ <br> $Var\left({}_tL^g \mid T_x \geq t\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_{x+t} - \left(\bar{A}_{x+t}\right)^2\right)$ | $_kL^g = \left(b+E+\frac{G-e}{d}\right)Z - \frac{G-e}{d}$ <br> $E\left[{}_kL^g \mid K_x \geq k\right] = (b+E)A_{x+k} - (G-e)\ddot{a}_{x+k}$ <br> $Var\left({}_kL^g \mid K_x \geq k\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_{x+k} - \left(A_{x+k}\right)^2\right)$ |
| **Fully continuous n-year endowment insurance** | **Fully discrete n-year endowment insurance** |
| $_tL^g = \begin{cases} (b+E)\,v^{T_{x+t}} - (G-e)\left(\frac{1-v^{T_{x+t}}}{\delta}\right), & T_{x+t} < n-t \\ (b+E)\,v^{n-t} - (G-e)\left(\frac{1-v^{n-t}}{\delta}\right), & T_{x+t} \geq n-t \end{cases}$ | $_kL^g = \begin{cases} (b+E)\,v^{K_{x+k}+1} - (G-e)\left(\frac{1-v^{K_{x+k}+1}}{d}\right), & K_{x+k} = 0, \ldots, n-k-1 \\ (b+E)\,v^{n-k} - (G-e)\left(\frac{1-v^{n-k}}{d}\right), & K_{x+k} = n-k, n-k+1, \ldots \end{cases}$ |
| $_tL^g = \left(b+E+\frac{G-e}{\delta}\right)Z - \frac{G-e}{\delta}$ <br> $E\left[{}_tL^g \mid T_x \geq t\right] = (b+E)\bar{A}_{x+t:\overline{n-t}\rvert} - (G-e)\bar{a}_{x+t:\overline{n-t}\rvert}$ <br> $Var\left({}_tL^g \mid T_x \geq t\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_{x+t:\overline{n-t}\rvert} - \left(\bar{A}_{x+t:\overline{n-t}\rvert}\right)^2\right)$ | $_kL^g = \left(b+E+\frac{G-e}{d}\right)Z - \frac{G-e}{d}$ <br> $E\left[{}_kL^g \mid K_x \geq k\right] = (b+E)A_{x+k:\overline{n-k}\rvert} - (G-e)\ddot{a}_{x+k:\overline{n-k}\rvert}$ <br> $Var\left({}_kL^g \mid K_x \geq k\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_{x+k:\overline{n-k}\rvert} - \left(A_{x+k:\overline{n-k}\rvert}\right)^2\right)$ |

## RECURSIVE FORMULA

**Net Premium Policy Value:** $\left({}_kV + P\right)(1+i) = bq_{x+k} + {}_{k+1}V\ p_{x+k}$

**Gross Premium Policy Value:** $\left({}_kV^g + G - e\right)(1+i) = (b+E)\,q_{x+k} + {}_{k+1}V^g\ p_{x+k}$

**Interim Policy Value:** $\left({}_kV + P\right)(1+i_k)^s = b\ v^{1-s}\ {}_sq_{x+k} + {}_{k+s}V\ {}_sp_{x+k}$ where $0 < s < 1$.

$_{k+s}V\,(1+i_k)^{1-s} = b\ {}_{1-s}q_{x+k+s} + {}_{k+1}V\ {}_{1-s}p_{x+k+s}$ where $0 < s < 1$.

**For a special policy that pays $FA + {}_kV$ upon death:** $\left({}_{k-1}V + P\right)(1+i) = \left(FA + {}_kV\right)\ q_{x+k-1} + {}_kV\ p_{x+k-1}$

**Rearrange:** $_kV = ({}_{k-1}V + P)(1+i) - FA\, q_{x+k-1}$

**Obtain the formula:** $_kV = P\ddot{a}_{\overline{k}|}(1+i)^k - FA\sum_{j=1}^{k} q_{x+j-1}(1+i)^{k-j}$

## FPT RESERVES

**First year premium:** $E\left[{}_0L^{FPT}\right] = vq_x - \alpha = 0 \quad \rightarrow \quad \alpha = vq_x$

**Renewal year premium:**

$_1V^{FPT} = A_{x+1} - \beta\ddot{a}_{x+1} = 0 \quad \rightarrow \quad \beta = \frac{A_{x+1}}{\ddot{a}_{x+1}}$ whole life

$_1V^{FPT} = A^{\;1}_{x+1:\overline{n-1}|} - \beta\ddot{a}_{x+1:\overline{n-1}|} = 0 \quad \rightarrow \quad \beta = \frac{A^{\;1}_{x+1:\overline{n-1}|}}{\ddot{a}_{x+1:\overline{n-1}|}}$ n-year term

$_1V^{FPT} = A_{x+1:\overline{n-1}|} - \beta\ddot{a}_{x+1:\overline{n-1}|} = 0 \quad \rightarrow \quad \beta = \frac{A_{x+1:\overline{n-1}|}}{\ddot{a}_{x+1:\overline{n-1}|}}$ n-year endowment

## PROFIT TESTING

**Recall:** **Net premiums** are always calculated using the **equivalence principle**.

Equivalence principle → Net premiums are expected to cover benefits.

But **gross premiums** may not be calculated using the equivalence principle.

Equivalence principle → Gross premiums are expected to cover benefits and expenses.

Otherwise → Gross premiums are expected to cover benefits, expenses and profits (or losses).

**Reserve assumptions:** Assumptions imposed by government to determine reserves.

**Pricing assumptions:** Assumptions the company uses to determine premiums.

**Profit test assumptions:** Assumptions the company uses to conduct **profit testing**.

For traditional policies, assumptions are mortality (or decrements/transitions), interest rate, expenses, and lapse.

Premiums and reserves are pre-determined or "fixed".

**Define the following:**

$_tV$ is the reserve at time $t$

$G_t$ is the premium collected at time $t$

$E_t$ is the annual expenses incurred at time $t$

$i_t$ is the interest rate earned on the investments from time $t$ to time $t+1$

$DB_t$ is the death benefit payable at time $t$

$E_t^{DB}$ is the expenses associated with death benefit incurred at time $t$

**Expected profit:** $\Pr_{t+1} = ({}_tV + G_t - E_t)(1+i_t) - \left(DB_{t+t} + E_{t+1}^{DB}\right) q_{x+k} - {}_{t+1}V\, p_{x+t}$

This formula is for alive-dead model and can be modified for MS/MD/ML models.

## PROFIT MEASURES

**Profit vector:** $\Pr_k$ — $\Pr_0$ usually contains pre-contract expenses only.

**Profit signature:** $\Pi_k = {}_{k-1}p_x \Pr_k$ — Note that $\Pi_0 = \Pr_0$.

**Internal rate of return:** $IRR$ — Such that $\sum_{k=0}^{n} \frac{\Pi_k}{(1+IRR)^k} = 0$.

**Net present value**: $NPV = \sum_{k=0}^{n} \frac{\Pi_k}{(1+r)^k}$ Where $r$ is the hurdle rate.

**Partial NPV**: $NPV(t) = \sum_{k=0}^{t} \frac{\Pi_k}{(1+r)^k}$ Where $r$ is the hurdle rate.

**Discounted payback period**: Smallest $m$ Such that $NPV(m) = \sum_{k=0}^{m} \frac{\Pi_k}{(1+r)^k} \geq 0$.

**Profit margin**: $M = \frac{NPV}{EPV(\text{Premiums})}$ Where $EPV$ (Premiums) is calculated using $r$, the hurdle rate.

Premiums can be determined using a desired profit margin.

Reserves can be determined using a process called **zeroization**.

## ACTUAL PROFIT & GAIN BY SOURCE

**Expected profit**: Profit(Expected interest, Expected expense, Expected mortality)

**Actual profit**: Profit(Actual interest, Actual expense, Actual mortality)

**Overall gain**: Overall gain = Actual profit - Expected profit

The usual order of **gain by source**: Interest → Expense → Mortality

Gain from interest = Profit(A.interest, E.expense, E.mortality) − Profit(E.interest, E.expense, E.mortality)

Gain from expense = Profit(A.interest, A.expense, E.mortality) − Profit(A.interest, E.expense, E.mortality)

Gain from mortality = Profit(A.interest, A.expense, A.mortality) − Profit(A.interest, A.expense, E.mortality)

Overall gain = Gain due to interest + Gain due to expense + Gain due to mortality

## UNIVERSAL LIFE ACCOUNT VALUES

**Define the following:**

$AV_t$ is the account value at time $t$
$P_t$ is the premium collected at time $t$
$EC_t$ is the expense charge deducted from the account value at time $t$
$i_t^c$ is the credited interest rate earned on the investments time $t$ to time $t+1$
$CoI_t$ is the cost of insurance charge at time $t$
$q_{x+t}^*$ is the CoI mortality rate
$i^*$ is the CoI interest rate
$FA$ is the face amount
$SC_t$ is the surrender charge deducted from account value at time $t$

| | **Type A UL** | **Type B UL** |
|---|---|---|
| **Account Value** | $(AV_t + P_t - EC_t)(1+i_t^c) = \frac{1+i_t^c}{1+i^*} q_{x+t}^* FA + \left(1 - \frac{1+i_t^c}{1+i^*} q_{x+t}^*\right) AV_{t+1}$ | $(AV_t + P_t - EC_t)(1+i_t^c) = \frac{1+i_t^c}{1+i^*} q_{x+t}^* FA + AV_{t+1}$ |
| **Cost of Insurance** | $CoI_{t+1} = \frac{1}{1+i^*} q_{x+t}^* (FA - AV_{t+1})$ | $CoI_{t+1} = \frac{1}{1+i^*} q_{x+t}^* FA$ |
| Additional Death Benefit | $ADB_{t+1} = FA - AV_{t+1}$ | $ADB_{t+1} = FA$ |
| Death Benefit | $DB_{t+1} = FA$ | $DB_{t+1} = FA + AV_{t+1}$ |
| Surrender Benefit | $SB_t = AV_{t+1} - SC_{t+1}$ | $SB_t = AV_{t+1} - SC_{t+1}$ |

## UNIVERSAL LIFE CORRIDOR REQUIREMENT

**Corridor factor:** $\gamma$

| **Type A UL** | **Type B UL** |
|---|---|
| 1. Calculate $AV_{t+1}$ using the above formula.<br>2. If $\gamma AV_{t+1} \leq FA$, good.<br>3. If $\gamma AV_{t+1} > FA$, replace $FA$ with $\gamma AV_{t+1}$ in the formula and calculate $AV_{t+1}$ again. The revised death benefit is $\gamma AV_{t+1}$. | 1. Calculate $AV_{t+1}$ using the above formula.<br>2. If $\gamma AV_{t+1} \leq FA + AV_{t+1}$, good.<br>3. If $\gamma AV_{t+1} > FA + AV_{t+1}$, replace $FA + AV_{t+1}$ with $\gamma AV_{t+1}$ in the formula and calculate $AV_{t+1}$ again. The revised death benefit is $\gamma AV_{t+1}$. |

## UNIVERSAL LIFE PROFIT TESTING

**Define the following:** $E_t$ is the annual expenses incurred at time $t$

$E_t^{DB}$ is the expenses associated with death benefit incurred at time $t$

$E_t^{SB}$ is the expenses associated with surrender benefit incurred at time $t$

$i_t$ is the interest rate earned on the investment from time $t$ to time $t+1$

**Expected profit:** $\mathrm{Pr}_{t+1} = (AV_{t-1} + P_t - E_t)(1+i_t) - \left(DB_{t+1} + E_{t+1}^{DB}\right) q_{x+t}^{(d)} - \left(SB_{t+1} + E_{t+1}^{SB}\right) q_{x+t}^{(w)} - AV_{t+1} p_{x+t}^{(\tau)}$

# A4.6 Pension Plans and Retirement Benefits

## SALARY SCALE & REPLACEMENT RATIO

**Salary rate:** $\bar{S}_x$ — Salary rate at exact age $x$.

**Salary scale:** $s_x = \int_0^1 \bar{s}_{x+t} dt$ — Salary rate during the year $[x, x+1)$.

**Approximation:** $\bar{S}_x \approx S_{x-0.5}$

**Salary:** $S_y = S_x \frac{s_y}{s_x}$ — Salary received during the year $[y, y+1)$.

**Replacement ratio:** $R = \dfrac{\text{Pension income in the year after retirement}}{\text{Salary in the year before retirement}}$

**Define the following:** $xe$ is the age the member enters the plan

$xr$ is the age the member retires

$n_{xr} = xr - xe$ is the total number of years of service

$TPE_{xr} = S_{xe} + S_{xe+1} + \cdots + S_{xr-1}$ is the **total pensionable earnings**.

**Final average salary:** $S_{xr}^F = \dfrac{S_{xr-k} + \cdots + S_{xr-2} + S_{xr-1}}{k}$ — The average of salaries during the final $k$ years.

**Career average salary:** $S_{xr}^C = \dfrac{TPE_{xr}}{n_{xr}}$ — The average of salaries during the whole career.

## DEFINED BENEFIT PLANS

**Accrual rate of DB plan:** $\alpha$

**DB plan annual benefit:** $B = n_{xr}\alpha S^F_{xr}$ Final average salary

$B = n_{xr}\alpha S^C_{xr} = \alpha\, TPE_{xr}$ Career average salary

**Sponsor of DB plan:** If $B = n_{xr}\alpha S^F_{xr}$ is payable per year right after retirement and until the member dies, then the sponsor has to purchase an annuity worth $B\ddot{a}_{xr}$ for the member.
Early retirement benefits may be subject to a reduction factor $RF_{xr}$.

**APV of DB plan:**

$APV = \sum_{xr} n_{xr} \times \alpha \times S^F_{xr} \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Final average salary

$APV = \sum_{xr} \alpha \times TPE_{xr} \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Career average salary

**AL of DB plan, PUC method:**

$AL = \sum_{xr} n_x \times \alpha \times S^F_{xr} \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Final average salary

$AL = \sum_{xr} n_x \times \alpha \times \frac{TPE_{xr}}{n_{xr}} \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Career average salary

**AL of DB plan, TUC method:**

$AL = \sum_{xr} n_x \times \alpha \times S^F_x \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Final average salary

$AL = \sum_{xr} \alpha \times TPE_x \times (1 - RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Career average salary

**Normal contribution:** $C = v\ p^{00}_x\ {}_1V - {}_0V +$ EPV(Benefits for mid-year exits)

**No exit benefits, PUC method:** $C = {}_0V\left(\frac{1}{n}\right)$ No exit benefits, and ${}_tV$ calculated using PUC method.

**No exit benefits, TUC method:** $C = {}_0V\left(\frac{S_{x+1}}{S_x}\frac{n+1}{n} - 1\right)$ No exit benefits, and ${}_tV$ calculated using TUC method.

## RETIREE HEALTHCARE PLANS

**Premium for healthcare plan:** $B(x,t)$ Premium paid at time $t$ for a person aged $x$

**Benefit premium annuity:** $\ddot{a}_B(x,t) = \sum_{k=0}^{\infty} \frac{B(x+k,\ t+k)}{B(x,t)} v^k\, {}_kp_x$ This is an increasing annuity-due.

**Simplification:** $\ddot{a}_B(x,t) = \sum_{k=0}^{\infty} c^k(1+j)^k\ v^k\, {}_kp_x$ Where: $c = \frac{B(x+1,t)}{B(x,t)}$

$\ddot{a}_B(x,t) = \ddot{a}_{x@i^*}$ $1 + j = \frac{B(x,\ t+1)}{B(x,t)}$

$1 + i^* = \frac{1+i}{c(1+j)}$

**EPV of retiree healthcare plan:** $EPV = B(x,t)\ddot{a}_B(x,t)$

For someone retires at age $x$ and time $t$.

**AVTHB:**

$AVTHB = \sum_{t=0}^{65-x} B(x+t,t)\ddot{a}_B(x+t,t) \times v^t \times \frac{r_{x+t}}{l_x}$

$AVTHB = B(x,0) \sum_{t=0}^{65-x} \ddot{a}_{x+t@i^*} \times v^{*t} \times \frac{r_{x+t}}{l_x}$

Assume retirement must occur no later than age 65.

**APBO:** ${}_0V = \sum_{t=0}^{65-x} B(x+t,t)\ddot{a}_B(x+t,t) \times v^t \times \frac{r_{x+t}}{l_x} \times \frac{x-x_0}{x+t-x_0}$

Where $x_0$ is the age the employer begins working.

**Normal cost:** $C = vp_x\ {}_1V - {}_0V +$ EPV(Benefits for mid-year exits)

**No exit benefits.** $C = {}_0V(\frac{1}{n})$ Where n is the number of years of service