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A2.1 TIME VALUE OF MONEY

Present Value and **Future Value**: $FV = PV(1+i)^n$ $\quad PV = \dfrac{FV}{(1+i)^n}$

Accumulation and Amount Functions: $A(t) = Ka(t)$ $\quad A(0) = K$ $\quad a(0) = 1$

Effective Interest Rate: $i_t = \dfrac{a(t) - a(t-1)}{a(t-1)} = \dfrac{A(t) - A(t-1)}{A(t-1)}$

Simple Interest: $a(t) = 1 + it$ $\quad i_t = \dfrac{i}{1 + i(t-1)}$

Compound Interest: $a(t) = (1+i)^t$ $\quad i_t = i$

Nominal Rate of Interest:

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = 1 + i$$

$$i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right]$$

Effective Discount Rate: $d_t = \dfrac{a(t) - a(t-1)}{a(t)} = \dfrac{A(t) - A(t-1)}{A(t)}$

Discount Rate: $d = \dfrac{i}{1+i} = 1 - v = iv$ $\quad v = (1+i)^{-1}$ $\quad \dfrac{1}{d} - \dfrac{1}{i} = 1$

Nominal Rates of Discount:

$$\left(1 - \frac{d^{(m)}}{m}\right)^m = 1 - d = v$$

$$d^{(m)} = m\left[1 - (1-d)^{\frac{1}{m}}\right]$$

Force of Interest:

$$\delta_t = \frac{a'(t)}{a(t)} = \frac{A'(t)}{A(t)} = \frac{d}{dt}\ln a(t)$$

$$a(t) = e^{\int_0^t \delta_r dr}$$

Constant Force of Interest:

$$e^\delta = 1 + i$$

$$\delta = \ln(1+i)$$

PV of $1 Due in t Years:

$$PV = \frac{1}{a(t)} = (1+i)^{-t} = v^t = e^{-\delta t} = (1-d)^t$$

$$= \left(1 + \frac{i^{(m)}}{m}\right)^{-mt} = \left(1 - \frac{d^{(m)}}{m}\right)^{mt}$$

AV at time t of $1 invested at time 0:

$$AV = a(t) = (1+i)^t = e^{\delta t} = (1-d)^{-t}$$

$$= \left(1 + \frac{i^{(m)}}{m}\right)^{mt} = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt}$$

AV at time t_2 for $1 Invested at time t_1: $AV = \dfrac{a(t_2)}{a(t_1)} = e^{\int_{t_1}^{t_2} \delta_t dt}$

A2.2 ANNUITIES

Annuity-Immediate: (PV one period before first payment, AV at time of last payment)

$$PV = a_{\overline{n}|} = v + v^2 + \cdots + v^n = \frac{1-v^n}{i}$$

$$AV = s_{\overline{n}|} = 1 + (1+i) + \cdots + (1+i)^{n-1} = \frac{(1+i)^n - 1}{i} = a_{\overline{n}|}(1+i)^n$$

Annuity-Due: (PV at time of first payment, AV one period after last payment)

$$PV = \ddot{a}_{\overline{n}|} = 1 + v + \cdots + v^{n-1} = \frac{1-v^n}{d} = (1+i)a_{\overline{n}|} = \left(\frac{i}{d}\right)a_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

$$AV = \ddot{s}_{\overline{n}|} = (1+i) + (1+i)^2 + \cdots + (1+i)^n = \frac{(1+i)^n - 1}{d}$$

$$= (1+i)s_{\overline{n}|} = \left(\frac{i}{d}\right)s_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

Continuous Annuity:

$$PV = \bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta} = \left(\frac{i}{\delta}\right)a_{\overline{n}|} = \int_0^n v^t dt = \int_0^n e^{-\delta t}dt$$

$$AV = \bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta} = \left(\frac{i}{\delta}\right)s_{\overline{n}|} = \int_0^n (1+i)^{n-t}dt = \int_0^n e^{\delta(n-t)}dt$$

Deferred Annuity:

$$_{m|}a_{\overline{n}|} = {}_{m+1|}\ddot{a}_{\overline{n}|} = v^m a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$$

Perpetuity:

$$a_{\overline{\infty}|} = \frac{1}{i} \qquad \ddot{a}_{\overline{\infty}|} = \frac{1}{d} \qquad \bar{a}_{\overline{\infty}|} = \frac{1}{\delta}$$

Relationships:

$$a_{\overline{2n}|} = a_{\overline{n}|} + v^n a_{\overline{n}|} \qquad \frac{a_{\overline{2n}|}}{a_{\overline{n}|}} = 1 + v^n = \frac{\ddot{a}_{\overline{2n}|}}{\ddot{a}_{\overline{n}|}} \qquad \frac{a_{\overline{3n}|}}{a_{\overline{n}|}} = 1 + v^n + v^{2n}$$

Increasing Annuity - Payments in Arithmetic Progression:

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \qquad (Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{s_{\overline{n+1}|} - (n+1)}{i}$$

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} \qquad (I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{s_{\overline{n+1}|} - (n+1)}{d}$$

$$(I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta} \qquad (I\bar{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{\delta} = \frac{s_{\overline{n+1}|} - (n+1)}{\delta}$$

Decreasing Annuity - Payments in Arithmetic Progression:

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i} \qquad (Ds)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}$$

$$(D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d} \qquad (D\ddot{s})_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{d}$$

$$(D\bar{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{\delta} \qquad (D\bar{s})_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{\delta}$$

Increasing Perpetuity - Payments in Arithmetic Progression:

$$(Ia)_{\overline{\infty}|} = \frac{1}{i} + \frac{1}{i^2} = \frac{1}{id} \qquad (I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$$

***PQ* Formula for Arithmetic Annuities:** Payments are $P, (P+Q), (P+2Q), \ldots, (P+(n-1)Q)$

Annuities-immediate: $PV = P \cdot a_{\overline{n}|} + Q \cdot \left(\frac{a_{\overline{n}|} - n \cdot v^n}{i}\right)$ $\qquad FV = P \cdot s_{\overline{n}|} + Q \cdot \left(\frac{s_{\overline{n}|} - n}{i}\right)$

Annuities-due: $PV = P \cdot \ddot{a}_{\overline{n}|} + Q \cdot \left(\frac{a_{\overline{n}|} - nv^n}{d}\right)$ $\qquad FV = P \cdot \ddot{s}_{\overline{n}|} + Q \cdot \left(\frac{s_{\overline{n}|} - n}{d}\right)$

Perpetuities-immediate (infinite n): $PV = \frac{P}{i} + \frac{Q}{i^2}$

Perpetuities-due (infinite n): $PV = \frac{P}{d} + \frac{Q}{i \cdot d} = \left(\frac{P}{i} + \frac{Q}{i^2}\right) \cdot (1+i)$

m-thly Annuity:

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}} \qquad \ddot{a}_{\overline{n}|}^{(m)} = \frac{1-v^n}{d^{(m)}}$$

$$(Ia)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}} \qquad (I^{(m)}a)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|}^{(m)} - nv^n}{i^{(m)}}$$

Continuously Increasing Annuity:

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = \int_0^n tv^t dt = \int_0^n te^{-\delta t} dt$$

$$(\bar{I}\bar{s})_{\overline{n}|} = \frac{\bar{s}_{\overline{n}|} - n}{\delta} = \int_0^n t(1+i)^{n-t} dt = \int_0^n te^{\delta(n-t)} dt$$

Annuity with Varying Payments and Varying Force of Interest:

$$PV = \int_0^n f(t)e^{-\delta t} dt \qquad \text{If force of interest varies:} \qquad PV = \int_0^n f(t)e^{-\int_0^t \delta_r dr} dt$$

$$AV = \int_0^n f(t)e^{\delta(n-t)} dt \qquad \text{If force of interest varies:} \qquad FV = \int_0^n f(t)e^{\int_t^n \delta_r dr} dt$$

Geometric Annuity-Immediate: 1^{st} Payment is 1. Subsequent payments increase by a factor of $(1+k)$.

$$PV = \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i-k} \qquad FV = \frac{(1+i)^n - (1+k)^n}{i-k}$$

Geometric Annuity-Due: 1^{st} Payment is 1. Subsequent payments increase by a factor of $(1+k)$.

$$PV = \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{d-kv} \qquad FV = \frac{(1+i)^n - (1+k)^n}{d-kv}$$

Sum of a Geometric Series:

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$$

A2.3 LOANS

Amortization Method: Level payment P, Outstanding Balance B_t, Principal Repayment $Prin_t$

$$L = B_0 = Pa_{\overline{n}|} \qquad P = Prin_t + I_t \qquad I_t = iB_{t-1}$$

$$Prin_t = P - I_t = Pv^{n-t+1} \qquad Prin_{t+s} = Prin_t(1+i)^s \qquad I_t = P - Prin_t = P(1-v^{n-t+1})$$

$$\sum_{t=1}^n I_t = nP - L \qquad \sum_{t=1}^n Prin_t = L \qquad B_t = B_{t-1} - Prin_t$$

$B_t = Pa_{\overline{n-t}|}$ (Prospective) $\qquad B_t = L(1+i)^t - Ps_{\overline{t}|}$ (Retrospective)

A2.4 BONDS

Price Formulas: Number of coupon payments n, Coupon rate r, Face amount F, Maturity value C

$$P = Fra_{\overline{n}|} + Cv^n \qquad P = C + (Fr - Ci)a_{\overline{n}|}$$

$$P = K + \frac{g}{i}(C-K), \text{ where } K = Cv^n \text{ and } g = \frac{Fr}{C}$$

Callable Bonds: To calculate appropriate price:

If Bond is sold at a **Premium**, assume Earliest Redemption date

If Bond is sold at a **Discount**, assume Latest Redemption date

Exception: If a premium bond has a call premium, calculate price both ways (using Earliest Redemption date and using Latest Redemption date).

Use the smaller of the 2 calculated values.

If $g > i$, then $P > C$, and **Premium** $= P - C = (Fr - Ci)a_{\overline{n}|} = (Cg - Ci)a_{\overline{n}|}$

If $g < i$, then $P < C$, and **Discount** $= C - P = (Ci - Fr)a_{\overline{n}|} = (Ci - Cg)a_{\overline{n}|}$

Book Value at time t: $B_t = Fra_{\overline{n-t|}} + Cv^{n-t}$

Interest Earned: $I_t = iB_{t-1}$

If $g > i$, **Premium Amortized** at time $t = Fr - I_t = B_{t-1} - B_t = (Fr - Ci)v^{n-t+1}$

If $g < i$, **Discount Accumulated/Accrued** at time $t = I_t - Fr = B_t - B_{t-1} = (Ci - Fr)v^{n-t+1}$

A2.5 GENERAL CASH FLOW AND PORTFOLIOS

Internal Rate of Return: Given investment cash flows $C_0, C_1, \dots, C_n$, an internal rate of return is a solution for i in the equation:

$$C_0 + \frac{C_1}{(1+i)^1} + \frac{C_2}{(1+i)^2} + \cdots + \frac{C_n}{(1+i)^n} = 0 \quad \text{or} \quad C_0 + C_1 v + C_2 v^2 + \cdots + C_n v^n = 0$$

Net Present Value: $NPV = C_0 + \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + \cdots + \frac{C_n}{(1+i)^n}$

Time-weighted Rate of Return:

C'_k = cash flow at time t_k $\quad B'_k$ = fund balance at time t_k before C'_k occurs

j_k = effective rate earned over the period $[t_{k-1}, t_k]$ $\quad 1 + j_k = \frac{B'_k}{B'_{k-1} + C'_{k-1}}$

The time-weighted rate of return (j) for the period t_0 to t_m (usually 1 year) is found from the equation: $1 + j = (1 + j_1) \cdot (1 + j_2) \cdots (1 + j_m)$

Dollar-weighted Rate of Return:

A = Initial fund balance $\quad B$ = Final fund balance $\quad I$ = Interest earned

C_t = Cash flow at time t $\quad C = \sum C_t$ = Total (net) cash flows

$B = A + C + I \longrightarrow I = B - A - C$

The dollar-weighted rate of return (i) for a 1-year period is: $i = \dfrac{I}{A + \sum C_t(1-t)}$

Reinvestment Rates:

Interest rate i, Reinvestment rate i'

A single deposit of \$1, $AV = 1 + is_{\overline{n|}i'}$

Deposits of \$1 at beginning of each year, $AV = n + i(Is)_{\overline{n|}i'}$

Spot Rates:

Effective annual rate on investment for t years: r_t

Price of a t-year zero-coupon Bond: $P_t = (1 + r_t)^{-t}$

Forward Rates: (Annual Effective)

Forward rate for the period $(t, t+1)$: $f_{[t,t+1]} = \dfrac{(1 + r_{t+1})^{t+1}}{(1 + r_t)^t} - 1 = \dfrac{P_t}{P_{t+1}} - 1$

Forward rate for the period $(t, t+m)$: $(1 + r_t)^t (1 + f_{[t,t+m]})^m = (1 + r_{t+m})^{t+m}$

$$f_{[t,t+m]} = \left(\frac{(1 + r_{t+1})^{t+m}}{(1 + r_t)^t}\right)^{\frac{1}{m}} - 1$$

Inflation Rate:

Real rate of interest i', Inflation rate r

$1 + i = (1 + i')(1 + r)$ $\qquad i = i' + r + i'r$

Duration:

$$D_{\text{mac}}(i) = \frac{\sum tCF_t v^t}{\sum CF_t v^t} = -\frac{\frac{d}{d\delta}P}{P} \qquad D_{\text{mod}}(i) = \frac{\sum tCF_t v^{t+1}}{\sum CF_t v^t} = -\frac{\frac{d}{di}P}{P} = \frac{D_{\text{mac}}}{1+i}$$

Perpetuity: $D_{\text{mac}} = \dfrac{1+i}{i} = \dfrac{1}{d}$ $\qquad$ Mortgage or Level Annuity: $D_{\text{mac}} = \dfrac{(Ia)_{\overline{n|}}}{a_{\overline{n|}}}$

Bond: $D_{\text{mac}} = \dfrac{Fr(Ia)_{\overline{n|}} + nCv^n}{P}$ $\qquad$ Bond Sold at Par: $D_{\text{mac}} = \ddot{a}_{\overline{n|}}$

Convexity: $C_{\text{mac}} = \dfrac{\sum t^2 CF_t v^t}{\sum CF_t v^t} = \dfrac{\frac{d^2}{d\delta^2}P}{P}$ $\quad C_{\text{mod}} = \dfrac{\sum t(t+1) CF_t v^{t+2}}{\sum CF_t v^t} = \dfrac{\frac{d^2}{di^2}P}{P} = \dfrac{C_{\text{mac}} + D_{\text{mac}}}{(1+i)^2}$

Duration of a Portfolio: D_t and P_t are duration and price of t^{th} components of Portfolio

$$\text{D(Portfolio)} = \frac{D_1P_1 + D_2P_2 + \cdots + D_nP_n}{P_1 + P_2 + \cdots + P_n}$$

First-Order Modified Price Approximation: $P(i) \approx P(i_0) - D_{\text{mod}}(i_0)P(i_0)(i - i_0)$

First-Order Macaulay Price Approximation: $P(i) \approx P(i_0)\left(\dfrac{1+i_0}{1+i}\right)^{D_{\text{mac}}(i_0)}$

Second-Order Modified Price Approximation: $P(i) \approx P(i_0) - D_{\text{mod}}(i_0)P(i_0)(i - i_0) + C_{\text{mod}}(i_0)P(i_0)\dfrac{(i-i_0)^2}{2}$

A2.6 IMMUNIZATION

Requirements for Redington Immunization: If same interest rate applies to all CF's

(i) PV(Assets) = PV(Liabilities)	(i) $P_A = P_L$	(i) $\sum A_t v^t = \sum L_t v^t$
(ii) D_{mod}(Assets) = D_{mod}(Liabilities)	(ii) $P'_A = P'_L$	(ii) $\sum t A_t v^t = \sum t L_t v^t$
(iii) C_{mod}(Assets) > C_{mod}(Liabilities)	(iii) $P''_A > P''_L$	(iii) $\sum t^2 A_t v^t > \sum t^2 L_t v^t$

Full Immunization: (i) and (ii) as above, (iii) There is a single liability CF that is matched (in PV and D_{mod}) with 2 or more asset CFs, including at least one that occurs before and at least one that occurs after the liability CF.

Exact Matching or Dedication: Match both the amount and the time of Asset Cash Flows and Liability Cash Flows.

Dividend Discount Model: $P = \sum_{k=1}^{\infty} \dfrac{\text{Div}_k}{(1+i)^k}$

Stock: Dividend Growth Model: for valuing a stock, assuming that the dividends will increase by a factor of $(1+g)$ each period: $P = \frac{\text{Div}}{i-g}$

A2.7 INTEREST RATE SWAPS

Variables:

- t_k — end of k^{th} settlement period.
- n — number of settlement periods.
- Q_{t_k} — notional amount for k^{th} settlement period.
- R — swap rate (the *fixed* interest rate in a swap), expressed as an *effective rate per settlement period.*
- r_t — t-year spot rate, expressed as an annual effective rate.
- $f^*_{[t_{k-1},t_k]}$ — forward rate for the period from t_{k-1} to t_k (the k^{th} settlement period), expressed as an *effective rate per settlement period.*
- P_t — t-year present value factor (and the price of a zero-coupon bond maturing for 1 in t years).

Formulas:

$$P_t = (1+r_t)^{-t}$$

$$f^*_{[t_{k-1},t_k]} = \frac{(1+r_{t_k})^{t_k}}{(1+r_{t_{k-1}})^{t_{k-1}}} - 1 = \left(\frac{P_{t_{k-1}}}{P_{t_k}}\right) - 1$$

General formula for swap rate:

$$R = \frac{\sum_{k=1}^{n} (Q_{t_k} \cdot f^*_{[t_{k-1},t_k]} \cdot P_{t_k})}{\sum_{k=1}^{n} (Q_{t_k} \cdot P_{t_k})} = \frac{\sum_{k=1}^{n} \left(Q_{t_k} \cdot \left(P_{t_{k-1}} - P_{t_k}\right)\right)}{\sum_{k=1}^{n} (Q_{t_k} \cdot P_{t_k})}$$

For swaps with a level notional amount:

$$R = \frac{P_{t_0} - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}}$$

For non-deferred swaps with a level notional amount:

$$R = \frac{1 - P_{t_n}}{\sum_{k=1}^{n} P_{t_k}} \qquad (R \text{ is also the coupon rate for an } n\text{-period par coupon bond.})$$

Net Settlement Payment made by "Payer" to "Receiver":

$$(\text{notional amount}) \times (\text{fixed rate} - \text{variable rate})$$

Market Value of a Swap for the "Payer":

$$PV(\text{variable pmts to be received}) - PV(\text{fixed pmts to be paid})$$

$$= (\text{Recalculated swap rate} - \text{Original swap rate}) \cdot \sum_{k=1}^{n} (Q_{t_k} \cdot P_{t_k}),$$

where "$k = 1$ to n" represents the *remaining settlement periods.*

A2.8 DETERMINANTS OF INTEREST RATES

The following table shows the components of the loan interest rate for each situation we have described in the study manual. All rates are continuously compounded.

	interest rate with no default risk	interest rate with default risk
with no inflation	r	$R = r + s$
with a known rate of inflation	$r + i$	$R^* = r + s + i$
with an unknown rate of inflation ("nominal" int. rate)	$R_2 = r + i_e + i_u$	$R^* = r + s + i_e + i_u$
inflation-protected loan: contractual rate ("real" interest rate)	$R_1 = r - c$	$r + s - c$
inflation-protected loan: actual rate paid	$R_1^{(a)} = r - c + i_a$	$r + s - c + i_a$